Epsilon-Delta: a quick walkthrough

Calculus Ia, Fall 2019

Outline

1. Real numbers
2. Sequences and limits
3. Functions and limits
4. Limits with infinity
5. Continuous functions

Prerequisite: logic and symbols

How to write negations?

Consider the statement: all birds can fly. What is its negation?

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In mathematical term, if B is the set of birds and P is the property "can fly", the negation of

$$\forall x \in B, P(x)$$

is

$$\exists x \in B, \neg P(x).$$

This reads "some birds cannot fly".

In general one flips $\forall$ and $\exists$ and negate the conclusion.

How to write negations of implication?

Consider the phrase: if Alice is small then Alice is a rabbit. What is its negation?

In mathematical term, the negation of the implication

$$P \implies Q \qquad (\iff \neg P \vee Q)$$

is

$$P \wedge \neg Q$$

In the example above, the negation is “Alice is small, and Alice is not a rabbit”.

Exercise

Write the negation of

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One can use an intermediate step in the beginning.

$$\exists a \in S, \exists \epsilon > 0, \forall \delta > 0, \exists x \in S, \neg(|x - a| < \delta \implies |f(x) - L| < \epsilon).$$

Then expanding the negation of an implication,

$$\exists a \in S, \exists \epsilon > 0, \forall \delta > 0, \exists x \in S, |x - a| < \delta, |f(x) - L| \geq \epsilon.$$

How to give counter-examples?

We say that $f: I \to \mathbb{R}$ is continuous on an open interval I if

$$\forall a \in I, \forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x-a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Prove that the following function is not continuous on $(0, 1)$:

$$f(x) = 0 \quad \text{if} \quad 0 < x < \frac{1}{2}; \quad f(x) = 1 \quad \text{if} \quad \frac{1}{2} \leq x < 1.$$

Remember the negation before: we have to prove the following

$$\exists a \in (0,1), \exists \epsilon > 0, \forall \delta > 0, \exists x \in (0,1), |x-a| < \delta, |f(x) - f(a)| \geq \epsilon.$$

This means that we have to give the numbers a and ϵ first.

Example of a discontinuous function

Consider the function $f : (0, 1) \to \mathbb{R}$,

$$f(x) = 0 \quad \text{if} \quad 0 < x < \frac{1}{2}; \qquad f(x) = 1 \quad \text{if} \quad \frac{1}{2} \leq x < 1.$$

We prove that f is not continuous. Take $a = \frac{1}{2}$ and $\epsilon = \frac{1}{2}$.

Let $\delta > 0$ be arbitrary and prove that there exists an x in $(0, 1)$ such that $|x - \frac{1}{2}| < \delta$ and $|f(x) - f(a)| = |f(x) - 1| \geq \frac{1}{2}$.

Consider $x = \frac{1}{2} - \frac{\delta}{2}$. Then $|x - \frac{1}{2}| = \frac{\delta}{2} < \delta$. Since $0 < x < \frac{1}{2}$, $f(x) = 0$ and $|f(x) - f(a)| = |0 - 1| = 1 \geq \epsilon = \frac{1}{2}$.

This shows that f is not continuous on $(0, 1)$.

Remarks

How to prove $A \subset B$?

$$\forall x \in A, x \in B.$$

How to prove $\neg(A \subset B)$?

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Let X be a set and P, Q two properties. Consider

$$A = \{x \in X; P(x)\} \quad \text{and} \quad B = \{x \in X; Q(x)\}.$$

What does $A \subset B$ mean?

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Does $\forall x \in S, P(x)$ implies $\exists x \in S, P(x)$?

Yes if S is non-empty!

Real numbers

Upper bound and lower bound

Let S be a set of real numbers, i.e. $S \subset \mathbb{R}$. We say that S is

- bounded from above if there exists a real number c such that $x \leq c$ for all $x \in S$, i.e.
$$\exists c \in \mathbb{R}, \forall x \in S, x \leq c;$$
- bounded from below if there exists a real number d such taht $d \leq x$ for all $x \in S$, i.e.
$$\exists d \in \mathbb{R}, \forall x \in S, d \leq x.$$

Then c (resp. d) is called an upper bound (resp. a lower bound) of the set S.

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Similarly, for a real-valued function $f : X \to \mathbb{R}$ defined on any set X, we say that f is bounded (resp. bounded from above, bounded from below) if and only if $f(X)$ is bounded (resp. bounded from above, bounded from below) as a subset of $\mathbb{R}$, e.g.

$$\exists c \in \mathbb{R}, \forall x \in S, f(x) \leq c.$$

Theorem (Completeness axiom)

Let S be a non-empty subset of $\mathbb{R}$. If S is bounded from above then S has a least upper bound. If S is bounded from below then S has a greatest lower bound.

We have seen that we can replace “a” by “the”.

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If b is the least upper bound of the set S then for every $\epsilon > 0$, $b - \epsilon$ is not an upper bound and $b + \epsilon$ is not a least upper bound. In particular, for all $\epsilon > 0$, there exists an element $x \in S$ such that $b - \epsilon < x \leq b$, i.e.

$$\forall \epsilon > 0, \exists x \in S, b - \epsilon < x \leq b.$$

Theorem (Archimedean property)

If $a \in \mathbb{R}$ *is such that* $0 \leq a < \frac{1}{n}$ *for all* $n \in \mathbb{Z}_{>0}$*, then* $a = 0$.

Equivalently: there is no $b \in \mathbb{R}$ such that $n \leq b$ for all $n \in \mathbb{Z}_{>0}$.

Proof.

Consider the set $\mathbb{Z}_{>0}$. If such a real number $b \in \mathbb{R}$ exists, then $\mathbb{Z}_{>0}$ is bounded from above. Then by the completeness axiom, $\mathbb{Z}_{>0}$ has a least upper bound $C \in \mathbb{R}$.

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Consider the set $\mathbb{Z}_{>0}$. If such a real number $b \in \mathbb{R}$ exists, then $\mathbb{Z}_{>0}$ is bounded from above. Then by the completeness axiom, $\mathbb{Z}_{>0}$ has a least upper bound $C \in \mathbb{R}$.

No number smaller than C can be an upper bound for $\mathbb{Z}_{>0}$, and since $C - 1 < C$, there exists $n \in \mathbb{Z}_{>0}$ such that $C - 1 < n$. This implies $C < n + 1$ but $n + 1 \in \mathbb{Z}_{>0}$ if $n \in \mathbb{Z}_{>0}$. This is a contradiction since C is an upper bound for $\mathbb{Z}_{>0}$. $\square$

Sequences and limits

Limit of a sequence

Let $\{x_n\}$ be a sequence of real numbers. We say that the sequence converges if there exists an element $l \in \mathbb{R}$ such that, given $\epsilon > 0$, there exists a positive integer N such that for all $n \geq N$ we have $|x_n - l| < \epsilon$. To shorten,

$$\exists l \in \mathbb{R}, \forall \epsilon > 0, \exists N \in \mathbb{Z}_{>0}, \forall n \geq N, |x_n - l| < \epsilon.$$

The number l, if it exists, is unique.

Proof: uniqueness of the limit.

Suppose that there exists also $l' \in \mathbb{R}$ such that, given $\epsilon > 0$, there exists $N' \in \mathbb{Z}_{>0}$ such that for all $n \geq N'$ we have $|x_n - l'| < \epsilon$.

Consider $N_1 = \max(N, N') \in \mathbb{Z}_{>0}$. We deduce that for all $n \geq N_1$,

$$|l - l'| = |l - x_n + x_n - l'| \leq |x_n - l| + |x_n - l'| < \epsilon + \epsilon = 2\epsilon.$$

Since this is true for all $\epsilon > 0$, necessarily we have $l = l'$. $\square$

Boundedness of a convergent sequence

A real convergent sequence $\{x_n\}$ is bounded.

Proof.

Exercise. $\square$

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The converse statement (i.e. boundedness implies convergence) is false!!!

A statement is only equivalent to its contrapositive, it means that

$$P \implies Q \quad \text{if and only if} \quad \neg Q \implies \neg P.$$

In our case, the contrapositive is "an unbounded sequence is not convergent".

Example

The sequence $\{-n\}$ is not convergent.

Monotone convergence theorem

Let $\{x_n\}$ be an increasing real sequence, i.e. $\forall n \in \mathbb{Z}_{>0}$, $x_n \leq x_{n+1}$.

If $\{x_n\}$ is also bounded from above, then the least upper bound of the set $\{x_n\}$ is the limit of the sequence (in particular it converges in $\mathbb{R}$).

Proof.

Let $b = \sup\{x_n\}$. Let $\epsilon > 0$.

Since $b - \epsilon$ is not an upper bound of the set $\{x_n\}$, there exists some $N \in \mathbb{Z}_{>0}$ such that $b - \epsilon < x_N \leq b$.

Since $\{x_n\}$ is increasing, for all $n \geq N$, $b - \epsilon < x_N \leq x_n \leq b$.

It follows that for all $n \geq N$, $|x_n - b| < \epsilon$.

Thus the sequence $\{x_n\}$ is converging and $b = \sup\{x_n\}$ is its limit. □

Example: the number e

The sequence $\{u_n\}_{n>0}$ with

$$u_n = \sum_{k=1}^{n} \frac{1}{k!} = \sum_{k=1}^{n} \frac{1}{1 \times 2 \times \cdots \times k}$$

is increasing and bounded from above (similar proof as $\zeta(2)$).

By the monotone convergence theorem, it converges (towards $\sup\{u_n\}$).

Its limit is denoted e.

Example: geometric series

Let $0 < q < 1$. The sequence $\{v_n\}$ with

$$v_n = \sum_{k=0}^{n-1} q^k$$

is increasing and converges to $\frac{1}{1-q}$.

We use the fact that

$$(1-q)v_n = 1 - q + q - q^2 + q^2 - q^3 + \cdots + q^{n-1} - q^n = 1 - q^n < 1.$$

and also, if $0 < q < 1$,

$$\lim_{n \to \infty} 1 - q^n = 1.$$

Point of accumulation

Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$ a number. We say that x is a point of accumulation of the sequence if given $\epsilon > 0$, there exist infinitely many integers $n \in \mathbb{Z}_{>0}$ such that $|x_n - x| < \epsilon$. Otherwise said,

$$\forall \epsilon > 0, \forall N \in \mathbb{Z}_{>0}, \exists n \geq N, |x_n - x| < \epsilon.$$

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Example

1. The sequence $\{(-1)^n\}$ has two accumulation points;
2. The sequence $\{n\}$ has no accumulation points;
3. The sequence $\{1, -\frac{1}{2}, 1, -\frac{1}{3}, 1, -\frac{1}{4}, \dots\}$ has two accumulation points;
4. If a sequence is convergent then its limit is a point of accumulation.

Bolzano-Weierstrass theorem, take I

If there exist $a, b \in \mathbb{R}$ such that $a \leq x_n \leq b$ for all $n \in \mathbb{Z}_{>0}$, then there exists a point of accumulation c of the sequence $\{x_n\}$ with $a \leq c \leq b$.

Proof.

For each $n > 0$, let $c_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. Then we have $c_n \leq c_{n+1}$: c_n is an increasing sequence. Since c_n is also bounded from above by b, the monotone convergence theorem applies: c_n converges to $c = \sup\{c_n\}$, and $a \leq c \leq b$.

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We prove that c_n is a point of accumulation of $\{x_n\}$. Since c_n converges to c,

$$\forall \epsilon > 0, \forall N \in \mathbb{Z}_{>0}, \exists m \ge N, |c_m - c| < \frac{\epsilon}{2}.$$

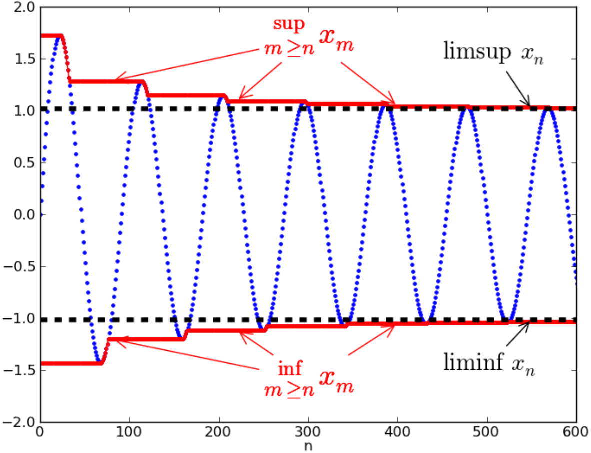
Also, since c_m is the infimum of the set $\{x_m, x_{m+1}, x_{m+2}, \ldots\}$,

$$(\forall \epsilon > 0, \forall m \in \mathbb{Z}_{>0},) \quad \exists k \ge m, |x_k - c_m| < \frac{\epsilon}{2}.$$

Together we get $(\forall \epsilon > 0,) \quad \exists k \ge N, |x_k - c| \le |x_k - c_m| + |c_m - c| < \epsilon.$ □

Extra: limit inferior

The number c in the previous proof is called the "limit inferior" of $\{x_n\}$, $c = \liminf_{n\to\infty}\{x_n\}$.



Bolzano-Weierstrass theorem, take II

Every bounded real sequence $\{x_n\}$ has a converging subsequence.

Proof.
The game here is to pick positive intergers $0 < n_1 < n_2 < \cdots < n_k < \dots$ such that the sequence $\{x_{n_k}\}_{k>0}$ converges. Let c be a point of accumulation of the sequence $\{x_n\}$.

First, pick n_1 such that $|x_{n_1} - c| < \frac{1}{1}$. Then we pick $n_2 > n_1$ such that $|x_{n_2} - c| < \frac{1}{2}$ etc. i.e. we successively pick $n_{k+1} > n_k$ such that $|x_{n_{k+1}} - c| < \frac{1}{k+1}$ for all $k \in \mathbb{Z}_{>0}$.

The subsequence $\{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$ converges to c since for all $\epsilon > 0$, there exists $m \in \mathbb{Z}_{>0}$ and that $\frac{1}{m} < \epsilon$, in such a way that for all $n_k \geq m$, $|x_{n_k} - c| < \frac{1}{n_k} \leq \frac{1}{m} < \epsilon$. □

A real sequence $\{x_n\}$ is said to be a Cauchy sequence if

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_{>0}, \forall m, n \geq N, |x_m - x_n| < \epsilon.$$

In $\mathbb{R}$, a sequence is convergent if and only if it is a Cauchy sequence.

Proof.

See exercise sheet. $\square$

Functions and limits

Adherence

Let $S \subset \mathbb{R}$ and $a \in \mathbb{R}$. We say that a is adherent to S if given $\epsilon > 0$, there exists an element $x \in S$ such that $|x - a| < \epsilon$. That is,

$$\forall \epsilon > 0, \exists x \in S, |x - a| < \epsilon.$$

If $a \in S$ then a is automatically adherent to S. The infimum and supremum of S, if they exist in $\mathbb{R}$, are adherent to S.

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Example

1. Let $S = (a, b) \subset \mathbb{R}$: the numbers a, b are adherent to S.
2. Let $S = [0, 1] \cup \{2\}$. Then 2 is adherent to S.
3. What is the difference between adherence and accumulation?

When a is adherent to a set $S \subset \mathbb{R}$ and f is a function $f : S \to \mathbb{R}$, we can investigate the question of limit of $f(x)$ as x approaches a (with respect to S a priori, but we will drop this soon).

Limit of a function

Let $a \in \mathbb{R}$ be adherent to $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ a function.

We say that the limit of $f(x)$ as x approaches a exists and is equal to L, if given $\epsilon > 0$, we can find $\delta > 0$ such that for all $x \in S$ satisfying $|x - a| < \delta$, we have $|f(x) - L| < \epsilon$. In short,

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in S, |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

When this is the case, we write

$$\lim_{x \to a, x \in S} f(x) = L.$$

In practise we can drop the dependence on S (admitted). The definition is technical but it is for quantifying the meaning of "approches": we cannot approach a point $a \in \mathbb{R}$ by elements of S if a is not "just right next" to S.

Slight variant of the definition

Let $a \in \mathbb{R}$ be adherent to $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ a function.

We say that the limit of $f(a+h)$ is L as h approaches 0 if

$$\forall \epsilon > 0, \exists \delta > 0, \forall |h| < \delta, a+h \in S \implies |f(a+h) - L| < \epsilon.$$

Example

Let $f(x) = c$ for all $x \in S$. Then

$$\lim_{x \to a} f(x) = c.$$

Indeed, given $\epsilon > 0$, for any $\delta > 0$ and any $x \in S$ we have $|f(x) - c| = 0 < \epsilon$.

A personal note: a touch of topology

From a topological point of view, we can rewrite the definition of the limit with the notion of neighborhood.

A set V is a neighborhood of a point $a \in \mathbb{R}$ if it contains an open interval $]a - \epsilon, a + \epsilon[$ for some $\epsilon > 0$. Then an equivalent definition of the limit of a function $f : \mathbb{R} \to \mathbb{R}$ at point a would be:

- If $L \in \mathbb{R}$ is such that for any neighborhood W of L, $f^{-1}(W)$ is a neighborhood of a, then L is the limit of f at point a.

In topology we resume this as "f is continuous at a point a if f^{-1} maps all neighborhoods of L to neighborhoods of a".

A personal note: continued

Now we rewrite everything with our usual $\epsilon - \delta$ definition. In particular

$$\exists L \in \mathbb{R}, \forall]L-\epsilon, L+\epsilon[, \exists]a-\delta, a+\delta[, \quad]a-\delta, a+\delta[\subset f^{-1}(]L-\epsilon, L+\epsilon[).$$

Making the conditions $\epsilon > 0$ and $\delta > 0$ explicit and using the definition of f^{-1},

$$\exists L \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \quad f(]a-\delta, a+\delta[) \subset]L-\epsilon, L+\epsilon[.$$

More precisely,

$$\exists L \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \quad x \in]a-\delta, a+\delta[\implies f(x) \in]L-\epsilon, L+\epsilon[.$$

More examples

When $a \in S$, the limit of $f(x)$ as x approaches a, if it exists, is automatically $f(a)$!

Indeed, for any $\epsilon > 0$, for all $\delta > 0$, $a \in S$ and $|a - a| = 0 < \delta$ so $|f(a) - L| < \epsilon$. This is true for all $\epsilon > 0$, we have necessarily $L = f(a)$.

Example

1. Let $f : [0, 1] \to \mathbb{R}$ such that $f(x) = x$. Then $\lim_{x \to 0} f(x) = 0$.
2. Let $g : [0, 1] \to \mathbb{R}$ such that $g(x) = x$ if $x > 0$ and $g(x) = 1$ if $x = 0$. Then the limit of g at 0 does not exist.
3. Let $h : (0, 1] \to \mathbb{R}$ such that $h(x) = x$. Then $\lim_{x \to 0} h(x) = 0$.
4. Let $i : (\frac{1}{2}, 1] \to \mathbb{R}$ such that $i(x) = x$. Then $\lim_{x \to 0} i(x)$ is not defined.

One sided limits

Let $I \subset \mathbb{R}$ be within the domain of definition of f, and $a \in I$ some point.

The right-sided limit can be rigorously defined as:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, x - a \in (0, \delta) \implies |f(x) - L| < \epsilon.$$

Similarly for the left-sided limit:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, x - a \in (-\delta, 0) \implies |f(x) - L| < \epsilon.$$

Example

1. $\lim_{x \to 0^+} \frac{x}{|x|} = 1$. $\lim_{x \to 0^-} \frac{x}{|x|} = -1$. $\lim_{x \to 0} \frac{x}{|x|}$ does not exist.
2. $\lim_{x \to 0^+} \frac{1}{e^{|\frac{1}{x}|}} = 0$. $\lim_{x \to 0^-} \frac{1}{e^{|\frac{1}{x}|}} = 0$. $\lim_{x \to 0} \frac{1}{e^{|\frac{1}{x}|}} = 0$.

Operations

Let $S \subset \mathbb{R}$, $a \in \mathbb{R}$ adherent to S and $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ functions. Suppose

$$\lim_{x \to a} f(x) = M, \quad \lim_{x \to a} g(x) = L.$$

Then

1. $\lim_{x \to a} (f + g)(x) = M + L$;
2. $\lim_{x \to a} (f \cdot g)(x) = M \cdot L$.

Example (Simple corollaries)

1. $\lim_{x \to a} (f - g)(x) = M - L$;
2. $\lim_{x \to a} (\lambda \cdot f)(x) = \lambda M, \quad \forall \lambda \in \mathbb{R}$;
3. If $M = 0$ or $L = 0$, then $\lim_{x \to a} (f \cdot g)(x) = 0$.

Proofs

Let $\epsilon > 0$. There exists a $\delta > 0$ such that for all $x \in]a-\delta, a+\delta[$,

$$|f(x) - M| < \epsilon; \quad |g(x) - L| < \epsilon.$$

For the sum, use $|f(x) + g(x) - M - L| \leq |f(x) - M| + |g(x) - L|$ so that for all $x \in]a-\delta, a+\delta[$ we have

$$|(f+g)(x) - (M+L)| < 2\epsilon.$$

For the product, use $|f(x) \cdot g(x) - M \cdot L| = |f(x) \cdot g(x) - f(x) \cdot L + f(x) \cdot L - M \cdot L| \leq |f(x)| \cdot |g(x) - L| + |f(x) - M| \cdot |L|$ so that for all $x \in]a-\delta, a+\delta[$ we have

$$|f(x) \cdot g(x) - M \cdot L| < \max(|M - \epsilon|, |M + \epsilon|)\epsilon + |L|\epsilon.$$

Comparaison of limits

Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$. Let a be adherent to S. Assume that $f(x) \le g(x)$ for x sufficient close to a (i.e. for all x in some neighborhood of a). Assume that

$$\lim_{x \to a} f(x) = L; \quad \lim_{x \to a} g(x) = M.$$

Then $L \le M$.

Proof.

Suppose that $L < M$ then $K = L - M < 0$.

Since $\lim_{x \to a}(f - g)(x) = L - M = K$, for $\epsilon = \frac{|K|}{2}$, there exists $\delta > 0$ such that

$$\forall x \in]a - \delta, a + \delta[, (f - g)(x) \in]K - \epsilon, K + \epsilon[\subset \mathbb{R}_{<0}.$$

This is impossible since $(f - g)(x) \ge 0$ for some x sufficiently close to a.

□

Be careful that the conclusion is always a large inequality!

Squeeze theorem

Let $S \subset \mathbb{R}$, a be adherent to S and $f, h, g : S \to \mathbb{R}$. Suppose that

$$f(x) \leq h(x) \leq g(x)$$

for x sufficiently close to a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x).$$

Then the limit of h at a exists and

$$\lim_{x \to a} h(x) = L.$$

Proof of the squeeze theorem

Let $\epsilon > 0$. There exists $\delta > 0$ such that for all $a - \delta < x < a + \delta$,

$$|f(x) - L| < \epsilon; \quad |g(x) - L| < \epsilon.$$

Consequently, for all $x \in]a - \delta, a + \delta[$,

$$|h(x) - L| \leq |h(x) - f(x)| + |f(x) - L| < (g(x) - f(x)) + \epsilon \leq |g(x) - L| + |f(x) - L| + \epsilon = 3\epsilon.$$

Example

Consider the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} \frac{1}{1+n^2x}.$$

Prove that $f(0) = 1$ and $f(x) = 0$ if $x > 0$.

Proof.

We seperate the cases where $x = 0$ and $x > 0$.

If $x = 0$, then for all $n > 0$, $\frac{1}{1+n^2x} = 1$: we have that $f(0) = 1$ as the limit of a constant sequence.

If $x > 0$, then we can use the following inequalites: $0 \leq \frac{1}{1+n^2x} \leq \frac{1}{x} \cdot \frac{1}{n^2}$. Since $\lim_{n\to\infty} \frac{1}{n^2} = 0$, by multiplication, $\lim_{n\to\infty} \frac{1}{x} \cdot \frac{1}{n^2} = 0$ and by the squeeze theorem, $\lim_{n\to\infty} \frac{1}{1+n^2x} = 0$. Thus $f(x) = 0$ if $x > 0$. $\square$

Extra problem for this week

Let A, B be two sets.

Suppose that there is an injection from A to B and an injection from B to A.

Prove that there is a bijection between A and B.

Interlude: $\pm\infty$

It is an abuse of notation, although frequently used, to write

$$\lim_{n\to\infty} x_n = \infty \quad (\text{or} \quad \lim_{n\to\infty} x_n = -\infty).$$

It means "x_n diverges to $+\infty$", and formally

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{Z}_{>0}, \forall n \geq N, x_n \geq M.$$

Similarly for functions.

Example

1. $\lim_{n\to\infty} -n = -\infty$;
2. $\lim_{x\to\infty} \ln(1+x) = \infty$;
3. $\lim_{n\to\infty} (-1)^n n$ does not exist.

Composition of functions

Let $S, T \subset \mathbb{R}$ and $a \in \mathbb{R}$ adherent to S, $b \in \mathbb{R}$ adherent to T. Let $f : S \to T$ and $g : T \to \mathbb{R}$ and assume that

$$\lim_{x \to a} f(x) = b; \quad \lim_{y \to b} g(y) = L.$$

Then

$$\lim_{x \to a} g(f(x)) = L.$$

Proof.

Let $\epsilon > 0$.

Since $\lim_{y \to b} g(y) = L, \exists \delta > 0, \forall y \in T, |y - b| < \delta \implies |g(y) - L| < \epsilon.$

Since $\lim_{x \to a} f(x) = b, \exists \delta_1 > 0, \forall x \in S, |x - a| < \delta_1 \implies |f(x) - b| < \delta.$

Combining we get $\exists \delta_1 > 0, \forall x \in S, |x - a| < \delta_1 \implies |g(f(x)) - L| < \epsilon.$ □

Example

Verify the following limits:

1. $\lim_{x \to \infty} e^{\cos\left(\frac{1}{x}\right)} = e$;
2. $\lim_{x \to \infty} \cos\left(e^{\frac{1}{x}}\right) = \cos(1)$;
3. $\lim_{x \to \infty} \cos\left(\frac{1}{e^x}\right) = 1$.

However $\lim_{x \to \infty} \cos(x)$ does not exist (exercise?).

Limits with infinity

From functions to sequences

Remember that a sequence $\{u_n\}$ can be seen as a function $u : \mathbb{Z}_{>0} \to \mathbb{R}$. The definition of the limit is very similar to that of functions, the difference being

$$\exists N \in \mathbb{Z}_{>0}, \forall n \geq N \quad \text{vs.} \quad \exists \delta > 0, \forall |x - a| < \delta.$$

The technical way of passing all the properties about limits of functions to limits of sequences is the following. Let $S = \{\frac{1}{n}; n > 0\}$ and consider the function $g : S \to \mathbb{R}$ defined by

$$\forall n > 0, g\left(\frac{1}{n}\right) = u_n.$$

One can check that

$$\lim_{x \to 0} g(x) = \lim_{n \to \infty} u_n.$$

Résumé of properties on limits of sequences

Using the change of variables $x = \frac{1}{n}$ as above we recover most properties of limits on real sequences. In particular,

- Results on addition, multiplication, comparison, composition of limits;
- Squeeze theorem.

Example

1. $\lim_{n\to\infty} n^{-3} = 0$ (since $\lim_{x\to 0^+} x^3 = 0$);
2. $\lim_{n\to\infty} \frac{n}{|n|} = 1$ (since $\lim_{x\to 0^+} \frac{|x|}{x} = 1$);
3. $\lim_{n\to\infty} \frac{\cos(n)}{n^2} = 0$ (squeeze theorem).

Limit at infinity

We can define the notion of limit at infinity for a function $f : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}$ where S contains arbitrary large numbers. Indeed, define for all $x \in S$, $x \neq 0$,

$$g\left(\frac{1}{x}\right) = f(x)$$

and declare that the limit of f at infinity exists if and only if the limit of g at 0 exists, and if they do, define

$$\lim_{x\to\infty} f(x) = \lim_{y\to 0} g(y).$$

Equivalently, we can also write $\lim_{x\to\infty} f(x) = L$ if

$$\forall \epsilon > 0, \exists B \in \mathbb{R}_{>0}, \forall x \in S, x \geq B \implies |f(x) - L| < \epsilon.$$

Example

Knowing that $\lim_{x\to\infty} e^x = +\infty$, $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} \ln(x) = \infty$, prove that

$$\lim_{x\to\infty} x^\alpha = L(\alpha)$$

where $L(\alpha) = 1$ if $\alpha = 0$, $L(\alpha) = +\infty$ if $\alpha > 0$ and $L(\alpha) = 0$ if $\alpha < 0$.

Proof.

Write $x^\alpha = e^{\alpha \ln(x)}$. If $\alpha = 0$ then it is constant equal to 1.

If $\alpha > 0$, then $\lim_{x\to\infty} \alpha \ln(x) = \infty$ and $\lim_{x\to\infty} e^{\alpha \ln(x)} = \infty$.

If $\alpha < 0$, then $\lim_{x\to\infty} \alpha \ln(x) = -\infty$ and $\lim_{x\to\infty} e^{\alpha \ln(x)} = 0$. □

Asymptotics and indeterminate forms

We will prove in the exercises the following comparaison results:

1. $$\lim_{x\to\infty} \frac{e^x}{x} = +\infty;$$
2. $$\lim_{x\to-\infty} xe^x = 0;$$
3. $$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0;$$
4. $$\lim_{x\to 0^+} x\ln(x) = 0.$$

Continuous functions

Continuity

Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ a function. Let $a \in S$. We say that f is continuous at a if $\lim_{x \to a} f(x)$ exists, and necessarily,

$$\lim_{x \to a} f(x) = f(a).$$

We say that f is continuous on S is f is continuous on every point $a \in S$.

Example

The usual functions (exponential, logarithm, polynomials, trigonometric functions) are continuous on their intervals of definition. The function $x \mapsto |x|$ is continuous on $\mathbb{R}$. The function $x \mapsto \frac{1}{x}$ is not continuous at 0.

Extension by continuity

Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ and $b \in \mathbb{R}$ adherent to S. A priori, f is not defined on b. But if $\lim_{x \to b} f(x)$ exists, we can extend f to a function

$$f : S \cup \{b\} \to \mathbb{R}$$

by defining $f(b) = \lim_{x \to b} f(x)$.

The extended function f is naturally continuous on $S \cup \{b\}$.

Example

1. The function $f(x) = x$ on $(0, 1)$ can be continuously extended to $[0, 1]$.
2. The function $g(x) = \frac{1}{x}$ on $(0, 1)$ can be continuously extended at 1, but not at 0.
3. The function $h(x) = \frac{\sin(x)}{x}$ on $(0, 1]$ can be continuously extended at 0 (by squeeze theorem!).

Operations

The sum, product and composition of continuous functions are continuous on the right domain of definition. Be careful in general with $x \mapsto \frac{1}{x}$, or quotients of functions.

We can also consider functions defined piece-wise. In this case we should examine the "glueing points" by comparing the two one-sided limits.

Example

1. The absolute value $x \mapsto |x|$ is continuous on $\mathbb{R}$. It is continuous on $\mathbb{R}_{>0}$ and on $\mathbb{R}_{<0}$ as elementary functions, and $\lim_{x\to 0^-} |x| = \lim_{x\to 0^+} |x|$ so that it is also continuous at the point 0.
2. The Heaviside function on $\mathbb{R}$ defined by $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$ is not continuous, since it is not continuous at the point 0.

Let f be a continuous function on a closed bounded interval $[a, b] \subset \mathbb{R}$. Then f is bounded and the bounds are attained.

Proof.

We first prove that f is bounded from above, say $f(x) \leq M \in \mathbb{R}$ for all $x \in [a, b]$. If this is not the case, we can find for each $n \in \mathbb{Z}_{>0}$ some element $x_n \in [a, b]$ such that $f(x_n) > n$. By Bolzano-Weierstrass, $\{x_n\}$ has an accumulation point x in $[a, b]$.

Extreme value theorem

Let f be a continuous function on a closed bounded interval $[a, b] \subset \mathbb{R}$. Then f is bounded and the bounds are attained.

Proof.

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Now $f(x) \in \mathbb{R}$ but also f is continuous at x so the value of f around x should not fluctuate too much: for every given $\epsilon > 0$ so in particular $\epsilon = 1$, there exists $\delta > 0$ such that for all $z \in]x - \delta, x + \delta[$, $|f(z) - f(x)| < 1$. This implies that for all $z \in]x - \delta, x + \delta[$, $f(z) < f(x) + 1$.

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In particular, for n large enough, there will be some $x_n \in]x - \delta, x + \delta[$ and $f(x_n) = n < f(x) + 1$. This is impossible since $\mathbb{Z}_{>0}$ is not bounded from above. □

Proof of Weierstrass' theorem continued

Proof continued.

We next prove that least upper bound β of f on $[a, b]$ is attained. By definition of supremum we can find for each $n \in \mathbb{Z}_{>0}$ some element $y_n \in [a, b]$ such that $\beta - f(y_n) = |f(y_n) - \beta| < \frac{1}{n}$. Again by Bolzano-Weierstrass $\{y_n\}$ has an accumulation point y in $[a, b]$.

Proof of Weierstrass' theorem continued

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Now $f(y) \leq \beta$ by definition of supremum and we prove that $f(y) = \beta$. Since f is continuous at y, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $z \in]y - \delta, y + \delta[$ then $|f(z) - f(y)| < \epsilon$.

Proof of Weierstrass' theorem continued

Proof continued.

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Now $f(y) \leq \beta$ by definition of supremum and we prove that $f(y) = \beta$. Since f is continuous at y, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $z \in]y - \delta, y + \delta[$ then $|f(z) - f(y)| < \epsilon$.

In particular for n large enough, some $y_n \in]y - \delta, y + \delta[$ satisfies both $|f(y_n) - f(y)| < \epsilon$ and $|f(y_n) - \beta| < \frac{1}{n}$. By triangular inequality we have $|f(y) - \beta| < \frac{1}{n} + \epsilon$. Take n large enough and ϵ small enough we conclude that necessarily $f(y) = \beta$. □

Combo: extension by continuity + Weierstrass

A priori, a function defined only on an open interval $(a, b) \subset \mathbb{R}$ does not enjoy the extreme value property: think of $x \mapsto \frac{1}{x}$ on $(0, 1)$ for instance.

However, if we can extend the function by continuity to both end points, then we can apply the Weierstrass theorem on the extended function (which is now defined on a closed interval).

Example

The function $u : x \mapsto \frac{\sin x}{x}$ defined on $(0, 1]$ is bounded. Indeed, it is continuous on $(0, 1]$ as compositions of elementary functions, and it has a limit at 0 since $\lim_{x \to 0^+} u(x) = 1$. We can extend it to a continuous function $\widetilde{u}$ on $[0, 1]$ defined as $\widetilde{u}(x) = u(x)$ on $(0, 1]$ and $\widetilde{u}(0) = 1$. By Weierstrass theorem, $\widetilde{u}$ is bounded on $[0, 1]$ and since u is the restriction of $\widetilde{u}$ to $(0, 1]$, the original function u is also bounded.

Intermediate value theorem

We sketch the idea of proof of the following version: if f is continuous on $[a, b] \subset \mathbb{R}$ and $f(a) > 0$, $f(b) < 0$ then there exists some $c \in (a, b)$ such that $f(c) = 0$.

Proof.

Consider the "first" instant when f crosses the zero value and prove that necessarily at this moment f takes the value 0. Let $S = \{x \in [a, b]; f(x) \leq 0\}$ and consider the instant $c = \inf(S)$ (since $S \neq \emptyset$ and bounded from below).

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Since c is adherent to S, $f(c) = \lim_{x \to c} f(x) \leq 0$ by comparison theorem.

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Since c is adherent to S, $f(c) = \lim_{x \to c} f(x) \leq 0$ by comparison theorem.

Consider also the set $T = \{x \in [a, b]; f(x) > 0\}$, non-empty and bounded from above. We claim that c is also adherent to T and by a similar argument as above we will have $f(c) \geq 0$ and this ends the proof. The claim is true since for all δ small enough, $c - \delta$ is not in S so necessarily $f(c - \delta) > 0$ so $c - \delta \in T$. $\square$

Inverse function of a continuous function on an interval

Remember that a function $f : S \to \mathbb{R}$ is strictly increasing if for all $x, y \in S$, $x < y \implies f(x) < f(y)$.

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. Then f establishes a bijection between $[a, b]$ and the interval $[f(a), f(b)]$.

Sketch of proof.

The proof is in three steps.

First, the image $f([a, b])$ is included in $[f(a), f(b)]$:

Then, the injectivity is given by the strict monotonicity.

Finally, the surjectivity is given by the intermediate value theorem. □

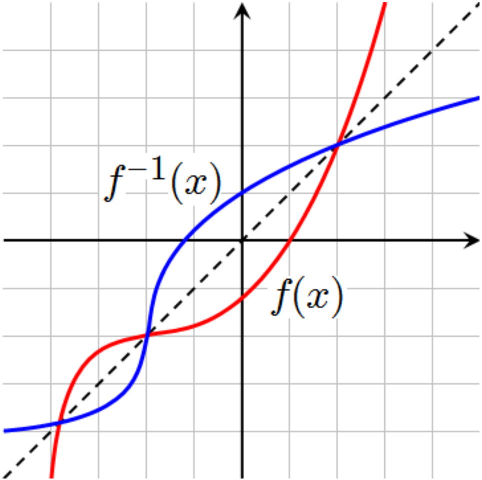
(Similar version works for f continuous and strictly decreasing!)

Continuity of the inverse function defined on an interval

In general, the inverse function of a continuous (bijective) function is not necessarily continuous.

However, the inverse function of a continuous (bijective) function $f : I \to J$ from an interval to another interval is continuous (on J).

We will come back to this later.



Practical guide: calculation of the inverse function

To calculate the inverse function, the safest way is to use the formula

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

when $f : A \to B$, $g : B \to C$ are both bijective.

In practice we can also solve the equation $f(x) = y \iff x = g(y)$. While doing it you are implicitly doing the above operations without giving names to all the functions appearing in the process.

Derivation

Definition

Let $I \subset \mathbb{R}$ be an interval (with more than one point) and $f : I \to \mathbb{R}$ be a function. We say that f is differentiable at $x \in I$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists in $\mathbb{R}$. We suppose $x + h \in I$ (if x is an end point of I, take one-sided limit).

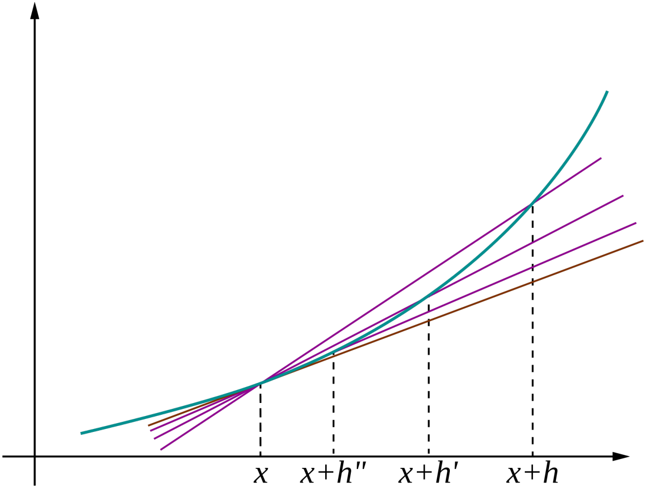
We denote the derivative of f at point x by $f'(x)$ (or $\frac{df(x)}{dx}$).

We admit in the following that

$$\forall x \in \mathbb{R}, \quad \frac{d\left(e^x\right)}{dx} = (e^x)' = e^x.$$

In particular, $x \mapsto \exp(x)$ is infinitely many times derivable, i.e. smooth.

Just because it's bad not to have a figure



Notations:

1. $\mathcal{C}^0$: continuous functions;
2. $\mathcal{D}^1$: derivable functions;
3. $\mathcal{C}^1$: derivable functions such that the derivative is continuous;
4. $\mathcal{C}^\infty$: infinitely derivable functions.

Theorem

Let $I \subset \mathbb{R}$ be an interval. A derivable function $f : I \to \mathbb{R}$ is continuous on I.

We get the inclusions $\mathcal{C}^\infty \subset \mathcal{C}^1 \subset \mathcal{D}^1 \subset \mathcal{C}^0$.

Let $I \subset \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$ two $\mathcal{D}^1$ functions. Let $x \in I$.

1. Derivative is a linear operation: $(f+g)'(x) = f'(x) + g'(x)$;
2. Product rule (Leibniz's rule): $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$;
3. Quotient rule: if $g(x) \neq 0$, then $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

For the composition of functions we have the chain rule.

Take two intervals $I, J \subset \mathbb{R}$ and suppose $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ such that $f(I) \subset J$ (the image of f is contained in the interval J), in such a way that we can consider the composition $g \circ f$. Then for $x \in I$,

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

Mnémotechnique

In case one forgets the quotient rule, remember that if $u(x) = \frac{1}{x}$ for $x \neq 0$,

$$u'(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

so that by the chain rule, if $g(x) \neq 0$,

$$\left(\frac{1}{g}\right)'(x) = (u \circ g)'(x) = -\frac{1}{g(x)^2} \cdot g'(x)$$

and by the Leibniz's rule, always if $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)'(x) = \left(f \cdot \frac{1}{g}\right)'(x) = f'(x)\frac{1}{g(x)} + f(x) \cdot \left(-\frac{g'(x)}{g(x)^2}\right).$$

For the composition, by abuse of notations, if $z = g(y)$ and $y = f(x)$,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

but pay attention to the term $g'(f(x))$ in the chain rule.

Example: the exponential function

1. We can recover the fact that $(\exp(x))' = \exp(x)$ for all $x \in \mathbb{R}$ if we only assume that $(\exp(0))' = 1$. Use the definition:

$$\lim_{h\to 0} \frac{\exp(x+h) - \exp(x)}{h} = \exp(x) \lim_{h\to 0} \frac{\exp(h) - 1}{h} = \exp(x) \cdot \exp'(0) = \exp(x).$$

2. We can calculate for example the derivative of $u(x) = \exp(\exp(x))$ by the chain rule: $u'(x) = \exp(\exp(x)) \cdot \exp(x)$.
3. Admitting the complex version $v(x) = \exp(ix)$, and admitting that $v'(x) = i\exp(ix)$, we recover formulæon sin and cos: remember that $\cos(x) = \Re(\exp(ix))$ and $\sin(x) = \Im(\exp(ix))$ (more on that later).

Spoiler: inverse function

Suppose a function f and its inverse f^{-1} are both derivable (we will see a theorem on that later), for now we just want a formula "à la physicienne".

Remember that $(f^{-1} \circ f)(x) = x$, and admitting that $(Id)'(x) = 1$, where $Id(x) = x$ for all $x \in \mathbb{R}$ is the identity map, the chain rule yields:

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

so that if $f'(x) \neq 0$, we get

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

or in other words when $f'(f^{-1}(y)) \neq 0$,

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Logarithm as the inverse of exponential

Remember that the natural logarithm $\ln : x \mapsto \ln(x)$, from $\mathbb{R}_{>0}$ to $\mathbb{R}$, is the inverse function of $\exp : \mathbb{R} \to \mathbb{R}_{>0}$. There will be a theorem that garantee that ln is derivable on $\mathbb{R}_{>0}$ (as the inverse function of the derivable function exp), but let's admit it now and see what formulæ can we get!

First, using the previous slide we can find $\ln'(x) = \frac{1}{x}$ for $x > 0$:

$$\ln'(x) = \left(\exp^{-1}\right)'(x) = \frac{1}{(\exp)'\left(\exp^{-1}(x)\right)} = \frac{1}{\exp(\ln(x))} = \frac{1}{x}.$$

In the middle, it's the derivative of exp evaluated at the point $\exp^{-1}(x)$!

Then we can play with the chain rule to get

$$(Id)'(x) = (\exp \circ \ln)'(x) = \exp(\ln(x)) \cdot \ln'(x) = x \cdot \frac{1}{x} = 1.$$

Elementary functions and their derivatives

Generalizing the previous slide, we have if $u(x) = x^\alpha$ for $\alpha \in \mathbb{R}$ then for $x > 0$,

$$u'(x) = (\exp(\alpha \ln))'(x) = (\alpha \ln)'(x) \cdot \exp(\alpha \ln)(x) = \alpha \cdot \frac{1}{x} \cdot x^\alpha = \alpha \cdot x^{\alpha - 1}.$$

In particular, if $x > 0$ and $v(x) = \frac{1}{x} = x^{-1}$ then

$$v'(x) = -\frac{1}{x^2}.$$

Also we get that for a polynomial $P(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0$ with $n \in \mathbb{Z}_{>0}$ and $a_n \neq 0$, we have for all $x \in \mathbb{R}$,

$$P'(x) = na_n \cdot x^{n-1} + (n-1)a_{n-1} \cdot x^{n-2} + \cdots + 2a_2 \cdot x + a_1.$$

Notice that $na_n \neq 0$ by assumption, so deriving a (non-constant) polynomial lowers its degree by 1.

Elementary functions and their derivatives: continued

Remember that $e^{ix} = \cos(x) + i\sin(x)$. Deriving (and identifying) we get

$$\cos'(x) = -\sin(x); \quad \sin'(x) = \cos(x).$$

Using the quotient rule (for $\cos(x) \neq 0$, i.e. when tan is well-defined),

$$\tan'(x) = \left(\frac{\sin}{\cos}\right)'(x) = 1 + \tan^2(x).$$

Remember also the hyperbolic trigonometric functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}; \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

As an exercise, verify that, with $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$,

$$\sinh' = \cosh; \quad \cosh' = \sinh; \quad \tanh' = 1 - \tanh^2 = \frac{1}{\cosh^2}.$$

Elementary functions and their derivatives: continued again

What about the inverse functions of these trigonometric functions?

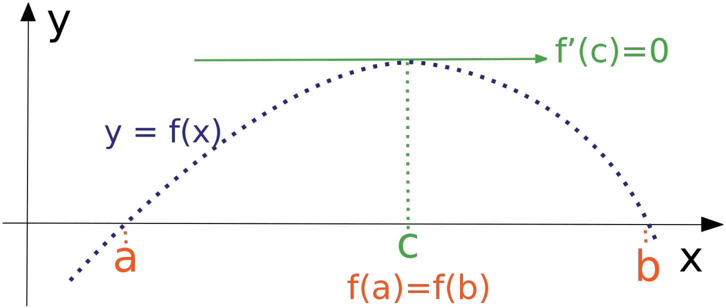
Applying the formula for the derivative of the inverse function we get for example:

1. $(\arcsin)'(x) = \frac{1}{\sqrt{1-x^2}}$;
2. $(\arccos)'(x) = -\frac{1}{\sqrt{1-x^2}}$;
3. $(\operatorname{arsinh})'(x) = \frac{1}{\sqrt{x^2+1}}$;
4. $(\operatorname{arcosh})'(x) = \frac{1}{\sqrt{x^2-1}}$.

See exercise sheet next week! For example with the physicists' notation:

$$\frac{d(arcsin(x))}{dx} = \frac{d\theta}{d\sin(\theta)} = \frac{d\theta}{\cos(\theta)d\theta} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-\sin^2(\theta)}} = \frac{1}{\sqrt{1-x^2}}.$$

Rolle's theorem



Mean value theorem