Elementary functions and their elementary properties

Calculus Ia, Fall 2019

Outline

1. Exponential function
2. (Natural) logarithm
3. Trigonometric functions
4. Polynomials
5. Hyperbolic functions

Exponential function

Definition

We first look at the function $\exp : \mathbb{R} \to \mathbb{R}$ defined as

$$\exp(x) = e^x.$$

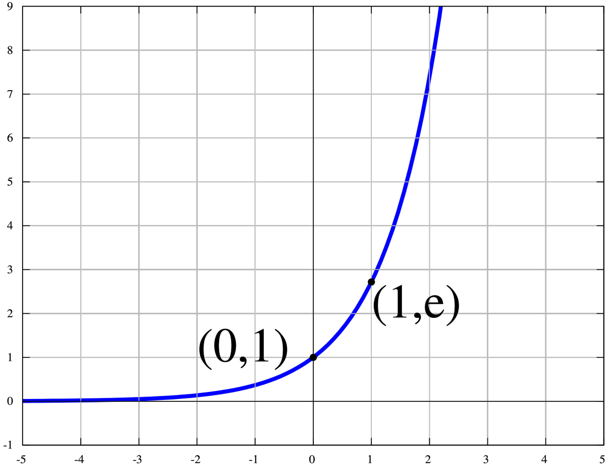
The characteristic relation of this function is

$$e^{x+y} = e^x \cdot e^y.$$

Remark: it is a group (iso)morphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_{>0}, \times)$. In plain language, it says that the expontial function transforms the structure of addition on $\mathbb{R}$ to the structure of multiplication on $\mathbb{R}_{>0}$.

Graph

Formally, the graph of a function $f : S \to \mathbb{R}$ is the subset $\{(x, f(x)); x \in S\}$ of $\mathbb{R}^2$.



Basic properties: generalities

All the elementary functions are smooth (i.e. infinitely derivable) on their respective domain of definitions. It follows that we can use all the tools on continuity and derivability to actually prove their basic properties.

We will use the exponential function as a first example. We study successively (in your prefered order) its parity, monotonicity, extrema, range among other things.

We need the fact that

$$\forall x \in \mathbb{R}, \quad (\exp)'(x) = \exp(x).$$

Parity

Let S be a subset of $\mathbb{R}$ and let $f : S \to \mathbb{R}$ be any function.

It is called

- even if for all $x \in S$, $f(x) = f(-x)$;
- odd if for all $x \in S$, $f(x) = -f(-x)$.

Notice that this condition imposes that the domain of definition S is symmetric with respect to 0!

We verify that the exponential function is neither even nor odd.

Development: symmetrization

From any function $f : \mathbb{R} \to \mathbb{R}$ there is a canonical way of decomposing it into the sum of an even and an odd function, that is

$$f = g + h$$

with $g : \mathbb{R} \to \mathbb{R}$ even and $h : \mathbb{R} \to \mathbb{R}$ odd.

- Existence: define $g(x) = \frac{f(x)+f(-x)}{2}$ and $h(x) = \frac{f(x)-f(-x)}{2}$;
- Uniqueness: the only function that is even and odd at the same time is the constant function equal to 0.

Remark: this is a decomposition of (infinite dimensional) vector space!

For the exponential function $\exp : \mathbb{R} \to \mathbb{R}$, applying the decomposition above, we get that:

- its even part is
$$\cosh = \frac{e^x + e^{-x}}{2};$$
- its odd part is
$$\sinh = \frac{e^x - e^{-x}}{2}.$$

Verify quickly that cosh is odd, sinh is even and $\cosh + \sinh = \exp$.

Monotonicity

The standard procedure of studying the monotonicity of a derivable function is to study the sign of its derivative. This is a consequence of the mean value theorem!

Indeed if $f : [a, b] \to \mathbb{R}$ is continuous on the interval $[a, b]$ and derivable on the interval (a, b), then we know that their exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Now if for all $c \in (a, b)$ the sign of $f'(c)$ does not change then the sign of $f(b) - f(a)$ is the same: i.e. if $f' > 0$ on (a, b) then necessarily $f(b) > f(a)$.

Now applying the same procedure for any pair $a', b' \in [a, b]$ we get a result of monotonicity.

Combo: intermediate value property

It is important to notice that the above only applies to domains which has the intermediate value property, i.e. intervals. For example, the function $x \mapsto x^{-1}$ has negative derivatives all the time but it is not monotone!

Suppose that the derivative of f on (a, b) is furthermore continuous. Suppose that the sign of f' changes on (a, b), then by Bolzano's theorem we know that there exists $d \in (a, b)$ such that $f'(d) = 0$.

Therefore in the case where f is derivable, f' is continuous on $[a, b]$ and $f' \neq 0$ on (a, b), then f is monotone on $[a, b]$. This is only a sufficient condition: the counter-example being the function $x \mapsto x^3$.

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Suppose that the sign of f' changes on (a, b), then by Darboux's theorem we know that there exists $d \in (a, b)$ such that $f'(d) = 0$.

Therefore in the case where f is derivable and $f' \neq 0$ on (a, b), then f is monotone on $[a, b]$. This is only a sufficient condition: the counter-example being the function $x \mapsto x^3$.

Combo: positivity

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on the interval $[a, b]$ and derivable on the interval (a, b). Suppose furthermore that $f(a) = 0$ and $f' \geq 0$ on (a, b).

Then f is increasing on $[a, b]$ and in particular, $f \geq 0$ on $[a, b]$.

Proof.

Let $z \in (a, b]$ and prove that $f(z) = f(z) - f(a) \geq 0$.

Since $z - a > 0$, $f(z) - f(a)$ has the same sign as $\frac{f(z)-f(a)}{z-a}$.

Since f is continuous on the interval $[a, z]$ and derivable on the interval (a, z), we know by mean value theorem that

$$\exists e \in (a, z), \quad \frac{f(z) - f(a)}{z - a} = f'(e).$$

But since $e \in (a, z) \subset (a, b)$ and $f' \geq 0$ on (a, b), $f'(e) \geq 0$ which implies $f(z) - f(a) \geq 0$, i.e. $f(z) \geq f(a) = 0$. □

Application: equality

A function is constant 0 if and only if it is positive and negative at the same time (try to write a decomposition of a function into its positive part and negative part as before, but it is not a decomposition of vector spaces!). In particular, the above observation can be used to show equality between functions. For example:

$$\forall x \in \mathbb{R}; \quad \cosh^2(x) - \sinh^2(x) = 1.$$

Proof.

Let $g(x) = \cosh^2(x) - \sinh^2(x) - 1$ and show that $g = 0$ on $\mathbb{R}$.

- First notice that $g(0) = 0$.
- For all $x \in \mathbb{R}$, $g'(x) = 2\cosh(x) \cdot \sinh(x) - 2\sinh(x) \cdot \cosh(x) = 0$.
- Since $\mathbb{R}$ is an interval and $0 \in \mathbb{R}$, we conclude that $g = 0$ for all $x \in \mathbb{R}$. $\square$

Application: inequality$^{(1)}$

The following inequality is super important and fundamental.

$$\forall x \in \mathbb{R}, \quad e^x \geq 1 + x.$$

Proof.

We will prove that the function $h(x) = e^x - 1 - x$ is positive on $\mathbb{R}$.

• First notice that $h(0) = 0$.

It is sufficient to prove that h is increasing on $\mathbb{R}_{\geq 0}$ and decreasing on $\mathbb{R}_{\leq 0}$.

• We the derivative of h: for all $x \in \mathbb{R}$, $h'(x) = e^x - 1$.

– We have $h' \geq 0$ on $\mathbb{R}_{\geq 0}$ so that for all $x \geq 0$, $h(x) \geq h(0) = 0$;
– We have $h' \leq 0$ on $\mathbb{R}_{\leq 0}$ so that for all $x \leq 0$, $h(x) \leq h(0) = 0$. □

We will see next a strategy of proof that does not depend on the monotonicity.

Combo: Weierstrass$^{(1)}$ and extrema

If $f : [a, b] \to \mathbb{R}$ is continuous (i.e. $\mathcal{C}^0$), recall that by Weierstrass theorem the extrema are necessarily attained when f is defined on a closed bounded interval).

From the above we also deduce that if $f : [a, b] \to \mathbb{R}$ is $\mathcal{C}^1$ then at any of its (local) extrema $e \in (a, b)$ we must have $f'(e) = 0$.

If $f' = 0$ has no solution in (a, b) then the extrema are necessarily $f(a)$ and $f(b)$.

Rolle's theorem relaxes the condition $\mathcal{C}^1$: it says that this argument also applies to $f : [a, b] \to \mathbb{R}$ that is continuous on $[a, b]$ and derivable on (a, b).

Combo: range of a continuous function

In general consider a function $f : I \to \mathbb{R}$ defined on any interval $I \subset \mathbb{R}$. If f is continuous, then by the intermediate value property, the range of f is also an interval.

In order to determine exactly the range $f(I)$, a general strategy is:

1. Study the supremum M and the infimum m of f on I: by the intermediate value property, the range $f(I)$ is one of the four following:
$$[m, M]; \quad [m, M[; \quad]m, M]; \quad]m, M[.$$
2. Study individually if M or m is attained by f on I and choose the right interval from above.

Range

The range $\exp(\mathbb{R})$ of the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is $]0, \infty[= \mathbb{R}_{>0}$.

• We first study the supremum and the infimum of the exponential function.

Since exp is $\mathcal{C}^1$, if $x \in \mathbb{R}$ is a (local) extremum then necessarily $(\exp)'(x) = \exp(x) = 0$. This equation has no solution in $\mathbb{R}$.

This means that the extrema can only be at infinities. We have

$$\lim_{x \to -\infty} e^x = 0; \quad \lim_{x \to \infty} e^x = \infty.$$

• We now test to see if 0 can be attained. Since the equation $e^x = 0$ has no solution in $\mathbb{R}$, 0 is never attained so the range of $\exp : \mathbb{R} \to \mathbb{R}$ is $]0, \infty[= \mathbb{R}_{>0}$.

Application: inequality$^{(2)}$

Let us write a different proof of the following inequality.

$$\forall x \in \mathbb{R}, \quad e^x \geq 1 + x.$$

Proof.

We will prove that the $\mathcal{C}^1$ function $h(x) = e^x - 1 - x$ is positive on $\mathbb{R}$.

This time our strategy will be to prove that $\inf(h) \geq 0$. Solving the equation $h'(a) = 0$ yields $a = 0$: we prove now that it is indeed a minimum.

Indeed since $\lim_{x\to\infty} h(x) = \lim_{x\to-\infty} h(x) = \infty$, by a generalization of Weierstrass' theorem h necessarily attains its minimum on $\mathbb{R}$ at some point b.

Since h is $\mathcal{C}^1$ on $\mathbb{R}$, by Rolle's theorem we have $h'(b) = 0$. Since a is the only solution to $h' = 0$, $b = a = 0$ and $\inf(h) = h(0) = 0$. □

Remark: we just proved that the range of h is $[h(0), \infty[= \mathbb{R}_{\geq 0}$.

Combo: Weierstrass' theorem$^{(2)}$

Suppose that $f : [a, b] \to \mathbb{R}$ is $\mathcal{C}^1$. This means its derivative is also continuous and we can apply Weierstrass' theorem to f': the result is known as mean inequality in some books. More precisely,

$$\exists m, M \geq 0, \quad m \leq \left| \frac{f(b) - f(a)}{b - a} \right| \leq M.$$

For example, one can take

$$m = \inf_{c \in [a,b]} |f'(c)|; \quad M = \sup_{c \in [a,b]} |f'(c)|.$$

Application: inequality$^{(3)}$

Let us apply the mean inequality to the exponential function on the interval $[0, x]$.

Let $x > 0$. Since exp is $\mathcal{C}^1$ on $[0, x]$, applying the mean inequality we have

$$1 = \inf_{t\in[0,x]} |\exp'(t)| \leq \left|\frac{\exp(x) - \exp(0)}{x - 0}\right| \leq \sup_{t\in[0,x]} |\exp'(t)| = \exp(x).$$

We get that $x \leq \exp(x) - 1 \leq x \exp(x)$, i.e.

$$\forall x > 0; \quad \exp(x) \geq 1 + x \text{ and } \exp(-x) \geq 1 - x.$$

Writing it differently,

$$\forall x \in \mathbb{R}, \quad e^x \geq 1 + x.$$

Recall that by definition:

$$1 = (\exp)'(0) = \lim_{x \to 0} \frac{e^x - \exp(0)}{x - 0} = \lim_{x \to 0} \frac{e^x - 1}{x}.$$

Just to make things interesting, let $x > 0$ be arbitrary and using mean inequality as before,

$$1 = \inf_{t \in [0,x]} |(\exp)'(t)| \leq \frac{e^x - e^0}{x - 0} = \frac{e^x - 1}{x} \leq \sup_{t \in [0,x]} |(\exp)'(t)| = e^x.$$

By the squeeze theorem, since $\lim_{x \to 0} 1 = \lim_{x \to 0} e^x$, we conclude that

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

exists and is equal to 1.

Asymptotics with exponential

We prove that

$$\lim_{x\to\infty} \frac{x}{e^x} = 0.$$

For example it suffices to prove that $2x^2 \leq e^x$ on $\mathbb{R}_{\geq 0}$. Let's prove

$$\forall x \geq 0, \quad e^x \geq 1 + x + \frac{x^2}{2}.$$

Let $h(x) = e^x - 1 - x - \frac{x^2}{2}$ and study its minimum on $\mathbb{R}_{\geq 0}$.

A solution to $h'(a) = 0$ verifies $\exp(a) - 1 - a = 0$.

Since $\exp(a) > 1 + a$ for $a > 0$, necessarily $a = 0$.

By an argument as before, $\inf_{x \geq 0} h(x) = h(0) = 0$ so $h(x) \geq 0$ on $\mathbb{R}_{\geq 0}$.

Strategies

Without actually doing the exercises, propose an attack plan.

1. Find the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{\exp(\cos(x)) - \exp(\cos(\frac{\pi}{2}))}{\cos(x) - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2}} \frac{\exp(\cos(x)) - 1}{\cos(x) - \frac{\pi}{2}}.$$

2. Prove that for all $x \leq 1$,

$$e^x \leq 1 + x + \frac{ex^2}{2}.$$

3. Let $0 < a < b$. Prove that for all $x > 0$,

$$ae^{-bx} - be^{-ax} > a - b.$$

1. Prove that for all $x \geq 0$,

$$\sinh(x) \geq x; \quad \cosh(x) \geq 1 + \frac{x^2}{2}.$$

2. Prove that for all $x \in [0, \frac{1}{2}]$,

$$e^{\frac{x^2}{2}} \geq \cosh(x) \geq e^{\frac{3x^2}{8}}.$$

3. Prove that for all $x \in \mathbb{R} \backslash \{0\}$,

$$\frac{e^x - 1}{x} \geq x + e - 2.$$

Recap'

The key of this chapter is the expression

$$\frac{f(b) - f(a)}{b - a}.$$

With $f : [a, b] \to \mathbb{R}$ a nice enough function (say $\mathcal{C}^1$), we can:

- Use mean value theorem to write the above as $f'(c)$ with $c \in (a, b)$;
- Use mean inequality to bound its absolute value;
- When $|b - a| \to 0$, rewrite it as the derivative $f'(a)$.

Development: Lipschitz function

Let I be a close bounded interval. A function $f : I \to \mathbb{R}$ is called Lipschitz if there exists some constant $C > 0$ such that

$$\forall a, b \in I, \quad \left| \frac{f(b) - f(a)}{b - a} \right| \leq C.$$

Every $\mathcal{C}^1$ function on I is Lipschitz with Lipschitz constant $\sup\limits_{t \in I} |f'(t)|$. This is a consequence of the mean inequality.

A common counter-example for a non-Lipschitz function is $x \mapsto x^{\frac{1}{2}}$ on $[0, 1]$.

Not so elementary things about the exponential function

Exponential function as the solution to a functional equation

The exponential function is the only continuous solution to the equation

$$\forall x, y \in \mathbb{R}, \quad f(x+y) = f(x) \cdot f(y)$$

with $f(1) = e$.

Idea of proof.

From the condition we can get the values for $f(2^{-n})$ for all $n \in \mathbb{Z}_{>0}$ by recurrence. For example, from

$$f(\frac{1}{2}) = (f(\frac{1}{4}))^2 \geq 0 \quad \text{and} \quad f(1) = (f(\frac{1}{2}))^2$$

we get that $f(\frac{1}{2}) = e^{\frac{1}{2}}$.

We then get the values for $f(m \cdot 2^{-n})$ for all $n, m \in \mathbb{Z}_{>0}$ and by continuity the value of f on $[0, 1]$. □

Exponential function as the solution to a differential equation

The exponential function is the only solution to the differential equation

$$\forall x \in \mathbb{R}, \quad f(x) = f'(x) \quad \text{and} \quad f(0) = 1.$$

This can be shown by using Picard-Lindelöf-Cauchy-Lipschitz condition.

To convince you....

First we verify that the exponential function solves the above equation.

Then to recover the result, write the differential equation as

$$\forall x \in \mathbb{R}, \quad (\ln \circ f)'(x) = 1 \quad \text{and} \quad (\ln \circ f)(0) = 0.$$

By one method before (cf. equality) we can show that $(\ln \circ f)(x) = x$ for all $x \in \mathbb{R}$, so that $f = \exp$. $\square$

Exponential function as a power series

We admit that

$$\forall x \in \mathbb{R}, \quad \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots.$$

A physicist proof.

Suppose the series on the right hand side is convergent for all $x \in \mathbb{R}$ (actually you can do it!).

Now if

$$\forall x \in \mathbb{R}, \quad h_k(x) = \frac{x^k}{k!}$$

then we verify that $(h_k)'(x) = h_{k-1}(x)$ for all $k \geq 1$.

It follows that the right hand side solves the differential equation

$$\forall x \in \mathbb{R}, \quad f(x) = f'(x) \quad \text{and} \quad f(0) = 1.$$

Natural logarithm

Definition

The natural logarithm function, denoted ln, is the inverse function to the exponential function. This means:

1. its domain of definition is the range of the exponential function, namely $\mathbb{R}_{>0}$;
2. its range is the domain of definition of the exponential function, namely $\mathbb{R}$;
3. For all $x \in \mathbb{R}_{>0}$, for all $y \in \mathbb{R}$,

$$\exp(\ln(x)) = x \quad \text{and} \quad \ln(\exp(y)) = y.$$

It follows that ln maps multiplication into addition:

$$\forall x, y \in \mathbb{R}_{>0}, \quad \ln(xy) = \ln(x) + \ln(y).$$

Monotonicity without using derivatives

Knowing that exp is strictly increasing and ln is its inverse function, we can deduce that ln is strictly increasing without calculating its derivative.

This is because if $x = \exp(a)$ and $y = \exp(b)$

$$a < b \iff \exp(a) < \exp(b)$$

is equivalent to

$$\ln(x) < \ln(y) \iff x < y.$$

Of course once we know that $(\ln)'(x) = \frac{1}{x}$ for all $x > 0$, we can recover the strictly monotonicity of ln.

Derivative as the inverse function

The inverse function of an $\mathcal{C}^1$ function on an interval is also $\mathcal{C}^1$.

Admitting this we can calculate the derivative of ln: from

$$\exp(\ln(x)) = x$$

we deduce by the chain rule:

$$(\ln)'(x) \times (\exp)'(\ln(x)) = 1.$$

If $(\exp)'(\ln(x)) \neq 0$, we often rewrite this as

$$(\ln)'(x) = \frac{1}{(\exp)'(\ln(x))} = \frac{1}{\exp(\ln(x))} = \frac{1}{x}.$$

We recover the fact that ln is strictly increasing on $\mathbb{R}_{>0}$.

Let us redo the study as the exponential functions, but a little differently.

Using the fact that $(\ln)'(1) = 1$ we deduce that

$$\lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{(1+x) - 1} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

Since the exponential function is continuous at 1,

$$\lim_{x\to 0} \exp\left(\frac{\ln(1+x)}{x}\right) = \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$$

From this we can deduce a formula of Euler:

$$\exp(z) = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n.$$

Other limits

Remember that $\exp(\alpha \ln(x)) = x^\alpha$ for $x > 0$.

If we want to calculate the following limit for $x > 0$:

$$\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha}$$

we have to study the derivative of $\alpha \mapsto x^\alpha$ at 0.

Applying chain rule we have

$$\frac{d \exp(\alpha \ln(x))}{d\alpha} = \ln(x) \cdot \exp(\alpha \ln(x))$$

and evaluating at $\alpha = 0$ yields the value $\ln(x)$.

We record that

$$\forall x > 0, \quad \lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \ln(x).$$

Some inequalities

Let's prove that

$$\forall x > 0, \quad \frac{x-1}{x} \leq \ln(x) \leq x - 1.$$

Or we can rewrite it as

$$\forall t > -1, \quad \frac{t}{1+t} \leq \ln(1+t) \leq t.$$

Sketch of proof.

We have already seen the right hand side. For the left hand side, we prove that

$$\forall x > 0, h(x) = x\ln(x) - (x-1) \geq 0.$$

We can show that $\inf_{x>0} h(x) = h(1) = 0$. □

Let's apply the mean inequality for ln on the interval $[t, t+1]$ with $t > 0$.

Since for all $t > 0$, ln is $\mathcal{C}^1$ on the interval $[t, t+1]$ and

$$\inf_{x\in[t,t+1]} \left|(\ln)'(x)\right| = \frac{1}{t+1}, \qquad \sup_{x\in[t,t+1]} \left|(\ln)'(x)\right| = \frac{1}{t},$$

we know by the mean inequality,

$$\forall t > 0, \quad \frac{1}{t+1} \leq \left|\frac{\ln(t+1) - \ln(t)}{(t+1) - t}\right| \leq \frac{1}{t}.$$

We record that

$$\forall t > 0, \quad \frac{1}{t+1} \leq \ln\left(1 + \frac{1}{t}\right) \leq \frac{1}{t}$$

or equivalently with $u = 1 + \frac{1}{t}$ or $t = \frac{1}{u-1}$,

$$\forall u > 0, \quad \frac{u-1}{u} \leq \ln(u) \leq u - 1.$$

Mean inequality again

Consider $x \mapsto \ln(\ln(x))$ on the interval $[t, t+1]$ with $t > 1$: it is a $\mathcal{C}^1$ function with derivative $x \mapsto \frac{1}{x\ln(x)}$ and the mean inequality yields

$$\frac{1}{(t+1)\cdot\ln(t+1)} \leq \ln(\ln(t+1)) - \ln(\ln(t)) \leq \frac{1}{t\cdot\ln(t)}.$$

From this we deduce that for $k \in \mathbb{Z}_{\geq 2}$,

$$\begin{aligned}\frac{1}{2\ln 2} + \cdots + \frac{1}{k\ln(k)} &\geq (\ln(\ln(3)) - \ln(\ln(2))) + \cdots + (\ln(\ln(k+1)) - \ln(\ln(k))) \\ &= \ln(\ln(k+1)) - \ln(\ln(2)).\end{aligned}$$

Since $\lim_{k\to\infty} \ln(\ln(k+1)) = \infty$, by the comparison theorem,

$$\sum_{k=2}^{\infty} \frac{1}{k\ln(k)} = \infty.$$

This is a special case of a common technique in integration.

Suppose we want to use the technique above to study the convergence of the series for $s > 0$:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

We have to interpret the term k^{-s} as $\sup_{x \in [t, t+h]} |f'(x)|$ for some suitable function f on some suitable interval $[t, t+h]$. For example a choice can be (for $s \neq 1$):

$$f(x) = \frac{x^{-s+1}}{-s+1} \quad \text{on} \quad [k, k+1].$$

In general this requires finding the primitive of some function.

Strategies

1. For all $x > 0$,
$$x - \frac{x^2}{2} < \ln(1 + x) < x.$$
2. For all $t > 1$,
$$\left(1 + \frac{1}{t}\right)^t < e < \left(1 + \frac{1}{t - 1}\right)^t.$$
3. For all $x \geq 0$ and $\alpha \geq 1$,
$$\ln(1 + x^\alpha) \leq \alpha \cdot x.$$

Sometimes it is handy to remember that

$$x \mapsto x \ln(x) - x$$

on $\mathbb{R}_{>0}$ is a primitive for $x \mapsto \ln(x)$.

Development: Euler–Mascheroni constant

The Euler-Mascheroni constant, denoted by γ, is defined as

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right).$$

Another way of writing it:

$$\gamma = \sum_{k=1}^{\infty} \left[\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right].$$

By some result above, we can see that each term in the sum is positive: by the monotone convergence theorem we only have to find an upper bound for this sum to conclude that it converges: another result does the job (which one)?

Development: logarithm as a power series

For $|z| < 1$, we can write

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k}.$$

The general rule for this kind of power series, provided that the series converges, is that for a smooth (i.e. $\mathcal{C}^\infty$) function defined on an interval $f : I \to \mathbb{R}$ and $a \in I$,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

As an exercise, try to calculate the k-th derivative of exp and ln and recover these power series (they are called Taylor series) without studying their convergences.

Development: intuitive Taylor-Lagrange theorem

To approach a function f by a constant at point a:

$$P_0(x) = f(a).$$

To approach a function f by a line at ponint a:

$$P_1(x) = f(a) + (x-a)f'(a).$$

To approach a function f by a second degree polynomial at point a:

$$P_2(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a).$$

Idea: if all the information you know about the function f is its derivatives at point a, to find the best polynomial P approaching f, you require that the derivatives of P are the same as f (as many as possible). The Taylor-Lagrange theorem gives information on the error of this approximation.

Trigonometric functions

Periodic functions

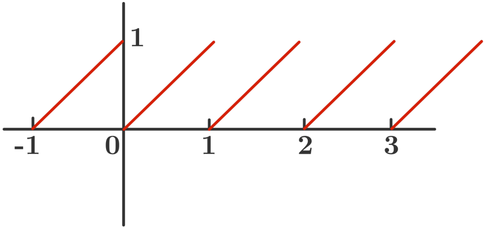
A function $f : S \to \mathbb{R}$ is called periodic of period $T > 0$ if

$$\forall x \in S, \quad f(x+T) = f(x).$$

In particular, the domain S itself must be periodic (and thus unbounded).

A periodic function is never injective, and to define a notion of inverse we always have to restrict the function to some smaller interval.

For example, the function $\{x\} = x - \lfloor x \rfloor$ is periodic of period 1 (or any $n \in \mathbb{Z}_{>0}$).



Weierstrass’ theorem strikes again

Let $f : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function.

We show that f is bounded: but Weierstrass’ theorem does not apply directly! Instead we show $f(\mathbb{R}) = f([0,1])$ and Weierstrass’ theorem applies to the latter.

It remains to show that $f(\mathbb{R}) \subset f([0,1])$ when f is 1-periodic.

Take an element $f(x) \in f(\mathbb{R})$ with $x \in \mathbb{R}$. Consider $\{x\} = x - \lfloor x \rfloor$: we have that $f(\{x\}) = f(x)$ by periodicity. Now $\{x\} \in [0,1]$ so that $f(x) = f(\{x\}) \in f([0,1])$.

Remark: the same proof works for any 2π-periodic function on $\mathbb{R}$.

sin and cos

Remember that for all $x \in \mathbb{R}$,

$$e^{ix} = \cos(x) + i \sin(x).$$

Since $x \mapsto e^{ix}$ is 2π-periodic, we have that sin and cos are 2π-periodic.

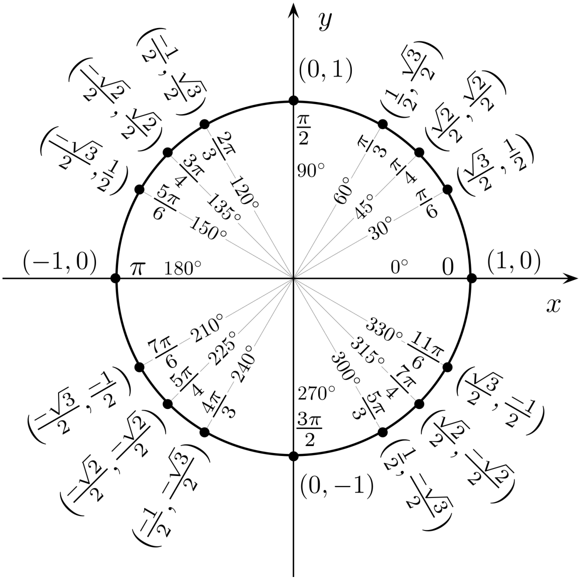
The Euler's identity

$$e^{i\pi} + 1 = 0$$

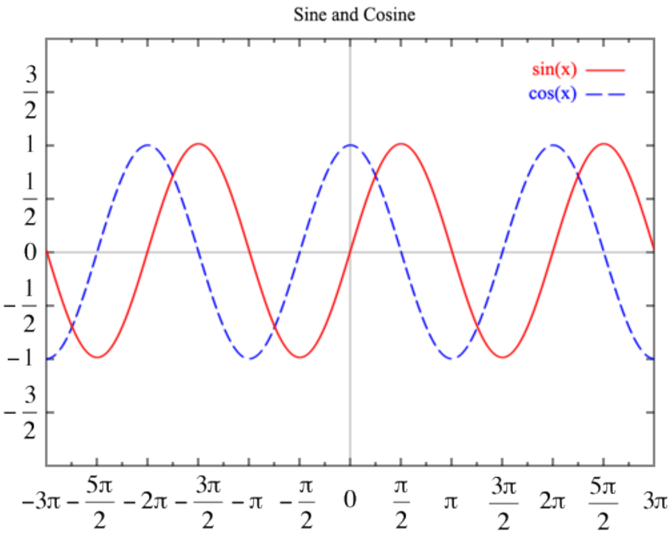
translates into

$$\cos(\pi) = -1, \quad \sin(\pi) = 0.$$

Pictures and special values



Pictures continued



Some identities

This pair of function is everywhere, especially in Fourier analysis. Example:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \, ;$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx;$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

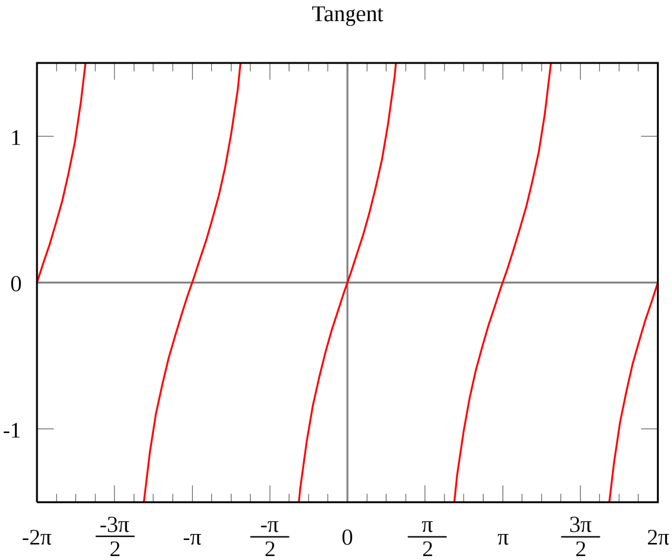
It is good to know something about $\sin(n\theta)$ and $\cos(n\theta)$ for $n \in \mathbb{Z}$. Recall from

$$e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}$$

we have by identifying the real and imaginary parts:

$$\sin(\alpha+\beta) = \sin(\alpha) \cdot \cos(\beta) + \sin(\beta) \cdot \cos(\alpha); \quad \cos(\alpha+\beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta).$$

Tangent function



Definition

The tangent function

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

is not defined if $x = k\pi + \frac{\pi}{2}$ for some $k \in \mathbb{Z}$.

It is periodic of period π, and the following asymptotics hold:

$$\lim_{x \to -\frac{\pi}{2}^+} \tan(x) = -\infty, \qquad \lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty.$$

We also verify that

$$\lim_{x \to 0} \frac{\tan(x)}{x} = 1.$$

Properties

The derivative of the tangent function has a nice expression:

$$(\tan)'(x) = 1 + (\tan)^2(x)$$

for $x \notin \{k\pi + \frac{\pi}{2}; k \in \mathbb{Z}\}$.

The sum of tangent has an ok-ish formula:

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}.$$

Inverse trigonometric functions

Definition

Recall the domain of definition and range of the inverse trigometric functions.

- $$\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$
- $$\arccos : [-1, 1] \to [0, \pi]$$
- $$\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

Remember the following (or calculate them quickly), valid on respective domains of definition.

- $$(\arcsin)'(z) = \frac{1}{\sqrt{1-z^2}}, \quad z \neq -1, 1$$
- $$(\arccos)'(z) = -\frac{1}{\sqrt{1-z^2}}, \quad z \neq -1, 1$$
- $$(\arctan)'(z) = \frac{1}{1+z^2}$$

Recall that $\sin' = \sqrt{1-\sin^2}$, $\cos' = -\sqrt{1-\cos^2}$ and $\tan' = 1 + \tan^2$.

When facing a complicated expression, one should always remember to check the domain of definition and image of the function: it is important to know where does an object "lives".

Example

$$\arcsin(\sin(2\pi)) = 0.$$

This is because the output of arcsin lives in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. To prove such identity, one should search for the unique solution $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(y) = \sin(2\pi)$.

In the case of trigonometric functions, using only the periodicity is sometimes not enough and one has to mix in some identites of the kind $\sin(x) = -\sin(\pi + x)$.

Polynomials

Definition

The study of polynomials is a very vast area of research.

We say that $P(x)$ is a real polynomial (function) of degree $n \geq 1$ if

$$P(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0 = \sum_{k=0}^{n} a_k \cdot x^k$$

with real coefficients $a_n, a_{n-1}, \ldots, a_0$ and $a_n \neq 0$.

Some terminologies:

- The constant polynomial $P(x) = c_0 \in \mathbb{R}$ is of degree 0 if $c_0 \neq 0$. By convention, it is of degree -1 or $-\infty$ if $c_0 = 0$.
- A polynomial is called unitary if its leading coefficient a_n is equal to 1.

Roots and their multiplicity

Let $P(x)$ be a polynomial. A solution α to the algebraic equation

$$P(x) = 0$$

is called a root of the polynomial P.

Sometimes a root can be multiple: the multiplicity of a root α is define as the integer $k \in \mathbb{Z}_{>0}$ such that

$$P(\alpha) = 0; \quad P'(\alpha) = 0; \quad \dots \quad P^{(k-1)}(\alpha) = 0; \quad P^{(k)} \neq 0.$$

Remember that for $P(x) = \sum_{k=0}^{n} a_k \cdot x^k$ of degree $n \geq 1$,

$$P'(x) = \sum_{k=0}^{n-1} (k+1) a_{k+1} \cdot x^k = n a_n \cdot x^{n-1} + (n-1) a_{n-1} \cdot x^{n-2} + \dots 2 a_2 \cdot x + a_1$$

is a polynomial of degree $n - 1$ since $n a_n \neq 0$ if $a_n \neq 0$.

Examples

1. The polynomial $P(x) = x$ has one simple root $\alpha = 0$. We verify that $P'(0) = 1 \neq 0$.
2. The polynomial $P(x) = (x-1)^2$ has one double root $\alpha = 1$. We verify that $P(1) = P'(1) = 0$ but $P''(1) = 2 \neq 0$.
3. The polynomial $P(x) = x^2 + 1$ has no real root: indeed, $P(x) > 0$ for all $x \in \mathbb{R}$. However, it has two complex roots: $P(i) = P(-i) = 0$. Notice that they are conjugate of each other.
4. Any odd degree real polynomial must have a real root: this is a consequence of Bolzano's theorem.

Fundamental theorem of algebra

We don't define the Euclidean division of polynomials in this course: the exercises concern mostly the analytic aspects of polynomials.

However, record the following theorem:

Theorem (Fundamental theorem of algebra)
Every non-constant single-variable polynomial with complex coefficients has at least one complex root.

All known algebraic proofs must use the intermediate value theorem. On Wikipedia there is a good overview of many different strategies of proof.

Using Rolle's theorem we can prove the following result:

Theorem

A real polynomial $P(x)$ of degree $n \geq 1$ has at most n distinct real roots.

Proof.

By induction: the case $n = 1$ is clear. Suppose $n \geq 2$.

Now suppose $P(x)$ has l different real roots $\alpha_1 < \cdots < \alpha_l$. Since $P(\alpha_1) = P(\alpha_2) = 0$, by Rolle's theorem there exists $\beta_1 \in (\alpha_1, \alpha_2)$ such that $P'(\beta_1) = 0$. Similarly we define $\beta_2, \ldots, \beta_{l-1}$.

Now $P'(x)$ has $l - 1$ distinct roots $\beta_1, \ldots, \beta_{l-1}$ and $P'(x)$ is a polynomial of degree $n - 1$. By the induction hypothesis, $l - 1 \leq n - 1$ so that $l \leq n$. □

We have a stronger version provided a stronger Rolle's theorem.

Theorem

A real polynomial $P(x)$ of degree $n \geq 1$ has at most n roots counted with their multiplicity.

For example, the root $\alpha = 1$ of the polynomial of degree $n = 3$

$$P(x) = (x - 1)^2 \cdot (x - 2)$$

counts twice, and the root $\beta = 2$ counts once: this polynomial has in total three roots counted with their multiplicity.

Example

We sketch the idea of proof using the following example:

$$P(x) = (x-1)^2 \cdot (x-2).$$

Knowing that the multiplicity of the root $\alpha_1 = 1$ is two and the multiplicity of the root $\alpha_2 = 2$ is one, we have that

$$P(\alpha_1) = P'(\alpha_1) = 0; \quad P(\alpha_2) = 0.$$

Using Rolle's theorem on $P(x)$ as before we can find $\beta_1 \in (\alpha_1, \alpha_2)$ such that $P'(\beta_1) = 0$. Using Rolle's theorem again, but this time on $P'(x)$, we can find $\gamma_1 \in (\alpha_1, \beta_1)$ such that $P''(\gamma_1) = 0$: this is because $P'(\alpha_1) = P'(\beta_1) = 0$.

Now to write a proof one has to change the way of arguing (exercise!).

Development: Taylor-Lagrange theorem

Remember the Taylor-Lagrange theorem: the idea is to approach a function by polynomials and give an explicit formula for the difference (i.e. error or rest term).

Rewrite the mean value theorem: if f is $\mathcal{C}^1$ on $[a, x]$ then

$$\exists t \in (a, x), \quad f(x) - f(a) = (b - a) f'(t).$$

Suppose f is $\mathcal{C}^\infty$ around the point a. The Taylor-Lagrange theorem generalizes the mean value theorem in the following way: if

$$P_k(x) = f(a) + (x - a) f'(a) + \cdots + \frac{(x-a)^k}{k!} f^{(k)}(a)$$

is the polynomial of order $k \geq 0$ approaching f near the point a, then

$$f(x) - P_k(x) = \frac{(x-a)^{k+1}}{k!} f^{(k+1)}(t)$$

for some t between x and a.

Development: Taylor-Lagrange theorem continued

Using Taylor-Lagrange theorem, we know that near 0,

$$\cos(x) = \cos(0) + (x - 0) \cdot \cos'(0) + \frac{(x-0)^2}{2} \cos''(t)$$

for some t in between 0 and x. This simplifies to

$$\cos(x) - 1 = -\frac{x^2}{2} \cos(t)$$

and as x goes to 0, $\cos(t)$ goes to 1 by continuity. So

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}.$$

Remark: a recent Fields medal celebrating the theory of regularity structure developed by Martin Hairer is similar in spirit to this kind of Taylor expansion.

Remainder: how do you prove the mean value theorem anyways?

Recall the Rolle's theorem for a smooth function: if $f(a) = f(x) = 0$ then there exists $t \in (a, b)$ such that $f'(t) = 0$.

Now what if $f(a) \neq f(x)$ but we still want to apply Rolle's theorem? We can consider a "drifted" version of f:

$$g(x) = f(a) - \lambda_1 \cdot (x - a).$$

Then if $g(a) = g(x) = 0$ and there exists $t \in (a, x)$ such that $g'(t) = 0$, i.e.

$$\lambda_1 = f'(t).$$

In some sense, we force the event that some function takes the same value at a and x to happen.

Remainder: how do you prove the mean value theorem anyways?

Recall the Rolle's theorem for a smooth function: if $f(a) = f(x) = 0$ then there exists $t \in (a, b)$ such that $f'(t) = 0$.

Now what if $f(a) \neq f(x)$ but we still want to apply Rolle's theorem? We can consider a "drifted" version of f:

$$g(x) = P_0(x) - \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

Then $g(a) = g(x) = 0$ and there exists $t \in (a, x)$ such that $g'(t) = 0$, i.e.

$$\frac{f(x) - f(a)}{x - a} = f'(t).$$

In some sense, we force the event that some function takes the same value at a and x to happen.

Development: Taylor-Lagrange theorem "proved"

In the higher order case, the "drift" should be a polynomial.

For $k = 1$, we have to prove

$$f(x) = f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2} \cdot f''(t) = P_1(x) + \frac{(x-a)^2}{2} \cdot f''(t)$$

for some t between a and x. We define

$$Q_1(x) = f(a) + (x-a) \cdot f'(a) + \lambda_2 \cdot (x-a)^2 = P_1(x) + \lambda_2 \cdot (x-a)^2$$

and look at $g_1(x) = f(x) - Q_1(x)$.

The correction term with λ_2 is there to force $g(a) = g(x)$ while preserving as many derivatives $g'(a)$ as possible: calculation yields

$$\lambda_2 = \frac{1}{(x-a)^2} \left(f(x) - P_1(x)\right).$$

Development: generalized Rolle's theorem

The function $g_1(x) = f(x) - Q_1(x)$ writes has the following property:

$$g_1(x) = 0; \quad g_1(a) = 0; \quad g_1'(a) = 0.$$

Now we know by Rolle's theorem there is some b between a and x such that

$$g_1'(b) = 0; \quad g_1'(a) = 0$$

and applying Rolle's theorem again, there is some t between a and b such that

$$g_1''(t) = 0.$$

This last equation is

$$f''(t) = Q_1''(t) = 2\lambda_2 = \frac{2}{(x-a)^2}\left(f(x) - P_1(x)\right).$$

Development: Landau’s little-o notation

Laudau’s little-o notation: we write for x near a point a,

$$f(x) \underset{x \to a}{=} o(g(x))$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

In other words, near the point a, $f(x) = g(x) \cdot \epsilon(x)$ with

$$\lim_{x \to a} \epsilon(x) = 0.$$

Example

When f is $\mathcal{C}^2$ near the point a we have

$$f(x) \underset{x \to a}{=} f(a) + (x - a) \cdot f'(a) + o((x - a)).$$

Development: Landau's big-O notation

Laudau's big-O notation: we write for x near a point a,

$$f(x) \underset{x \to a}{=} O(g(x))$$

if the following is bounded near a:

$$\left| \frac{f(x)}{g(x)} \right| \leq C.$$

In particular, if $f(x) \underset{x \to a}{=} o(g(x))$ then $f(x) \underset{x \to a}{=} O(g(x))$.

Example

When f is $\mathcal{C}^2$ near the point a we have

$$f(x) \underset{x \to a}{=} f(a) + (x - a) \cdot f'(a) + O((x - a)^2).$$

Hyperbolic geometry

Parallel postulate

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce (extend) a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance (radius).
4. That all right angles are equal to one another.
5. [The parallel postulate] That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

– «Elements», Euclide

Hyperbolic geometry

A commonly used reformulation of the parallel postulate is the Playfair's axiom:

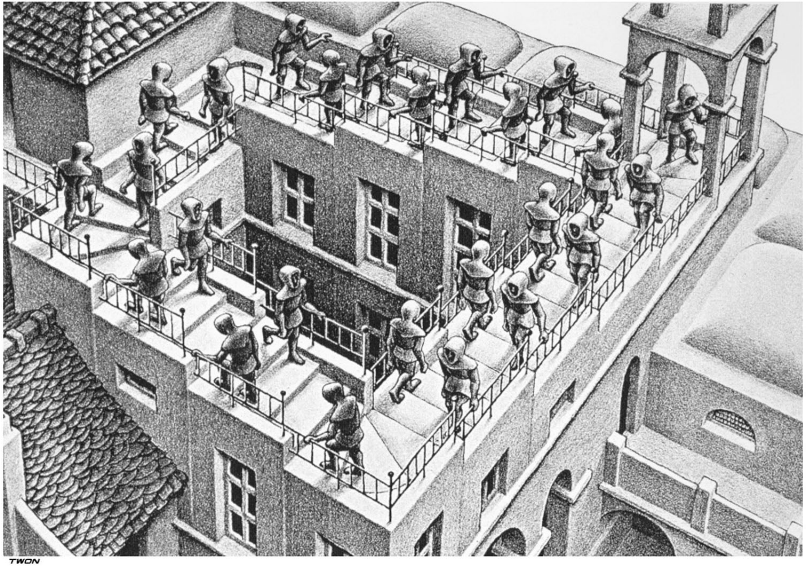
"In a plane, through a point not on a given straight line, at most one line can be drawn that never meets the given line."

In the hyperbolic model, within a two-dimensional plane, for any given line l and a point A, which is not on l, there are infinitely many lines through A that do not intersect l (Lobachevsky, Gauss, Bolyai).

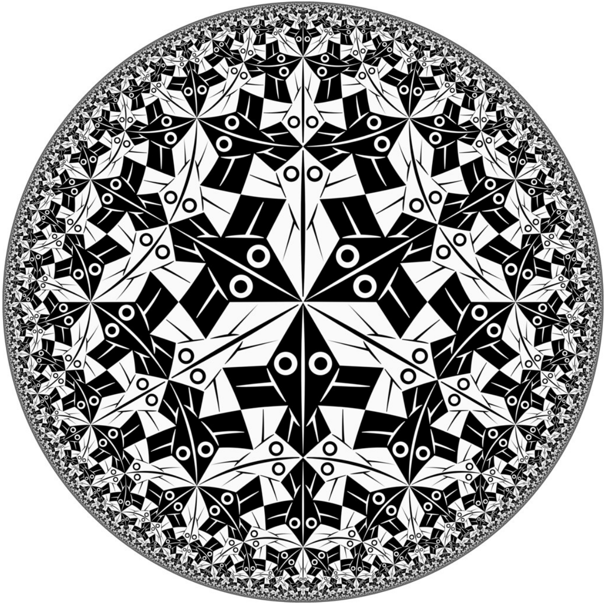
How can such a model exist? Beltrami said yes.

We are going to see an example known as the Poincaré's disk model.

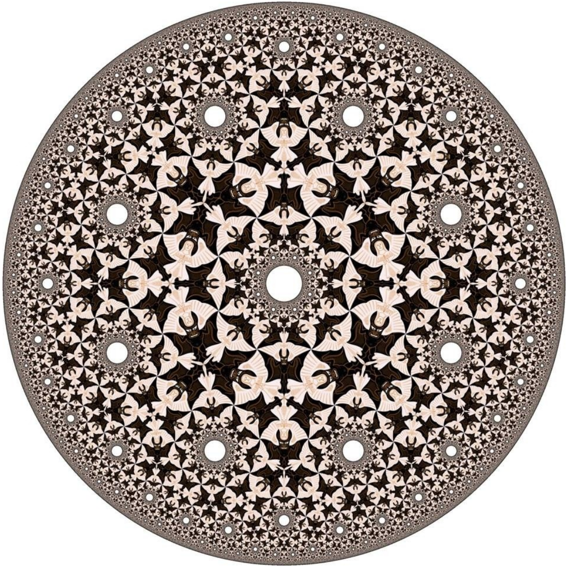
M. C. Escher: Ascending and Descending



M. C. Escher: Cercle Limit I



M. C. Escher: Cercle Limit IV



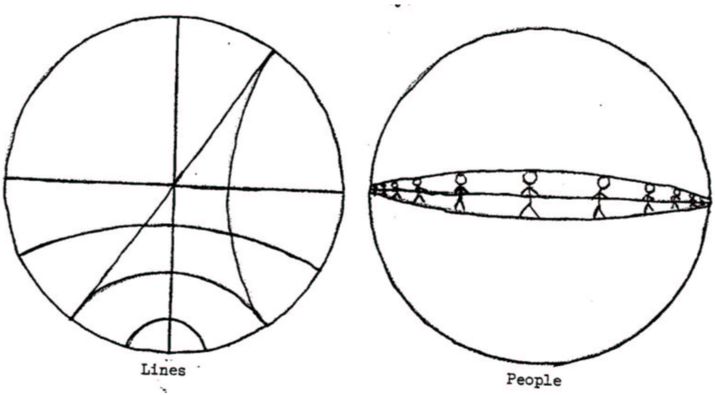
M. C. Escher: Cercle Limit III

Some good sources:

- M. C. Escher Collection. https://mcescher.com/gallery/mathematical/
- Gödel, Escher, Bach: An Eternal Golden Braid. http://geb.stenius.org
- The Poincaré disk model, animated. http://bulatov.org/math/1001/
- A hyperbolic computer game. http://roguetemple.com/z/hyper/
- Some mathematical reviews on Escher. Here and here and also here
- Some documentaries. BBC and Interview with Coxeter on Youtube

GEB is probably my personal favorite popular science book so far.

Poincaré’s disk model



By William P. Thurston.

Poincaré’s disk model

The first intuitive observation is the following for our ordinary Euclidean people:

1. The infinities are at the boundary: as you move to the boundary you are getting smaller and smaller and it is harder and harder to move. In other words, the hyperbolic metric (or distance) increases as you approach the boundary of the disk.
2. The hyperbolic straight lines (or geodesics) are not Euclidean straight lines. They are for us Euclidean poeple some “arcs of circle” that are orthogonal with the boundary. This stems from the fact that the distance is “distorted” near the boundary.
3. The sum of inner angles of a triangle is not necessarily π. In the extreme case it can be 0!

Hyperbolic metric: a first example

Consider the line $(0,1) \in \mathbb{D}$. For a point $r \in (0,1)$, its distance to 0 is defined as

$$\phi(0,r)\left(= 2\int_0^r \frac{dx}{1-|x|^2}\right) = \ln\frac{1+r}{1-r}.$$

But remember that

$$\ln\frac{1+r}{1-r} = 2\operatorname{artanh}(r)$$

so that we can also write

$$\phi(0,r) = 2\operatorname{artanh}(r).$$

Notice that

$$\lim_{r\to 1^-} \operatorname{artanh}(r) = \infty.$$

Artanh

The inverse hyperbolic tangent is defined on $(-1, 1)$ and maps to $(-\infty, \infty)$.
Its derivative writes, for $r \in (-1, 1)$,

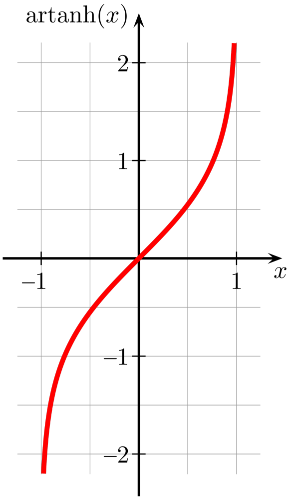
$$(\text{artanh})'(r) = \frac{1}{1 - r^2}.$$

Remember that

$$\tanh'(r) = 1 - \tanh^2(r).$$

We have the following limit:

$$\lim_{x \to 0} \frac{\text{artanh}(x)}{x} = 1.$$



Möbius transformations

The symmetry of the Poincaré disk is encoded in the so-called Möbius maps.

A Möbius map τ from $\mathbb{D} \to \mathbb{D}$ preserves angles:

$$\tau(z) = e^{i\varphi} \frac{z - z_0}{1 - \overline{z_0} z}$$

with $\varphi \in [0, 2\pi)$ and $z_0 \in \mathbb{D}$. Here, z_0 is mapped to 0 by τ and $e^{i\varphi}$ is a rotation.

You can also check that the boundary $\partial\mathbb{D}$ is mapped to itself under τ.

An important property that we will use: from any pair of points $u \neq v$ in $\mathbb{D}$, there exists a unique Möbius map τ such that

$$\tau(u, v) = (0, r)$$

with $r \in (0, 1)$. Indeed, we should choose in the above $z_0 = u$ and an angle φ such that the image of the second point is on the line $(0, 1)$.

Möbius transformations: examples

Consider the Möbius map τ with $z_0 = -\frac{1}{2}$ and $\varphi = 0$ (no rotation):

$$\tau(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z}.$$

We can calculate that

$$\tau\left(-\frac{1}{2}\right) = 0, \quad \tau(0) = \frac{1}{2}, \quad \tau\left(\frac{1}{2}\right) = \frac{4}{5} \dots$$

Geometrically, it "pushes" everything to the right side in a conformal manner.
Here is an animation.

Hyperbolic metric under Möbius transformations

We admit that the hyperbolic metric is invariant under Möbius maps: if ϕ is the hyperbolic metric and τ a Möbius map $\mathbb{D} \to \mathbb{D}$ then

$$\phi(z_1, z_2) = \phi(\tau(z_1), \tau(z_2)).$$

For example, take the τ before that maps 0 to $\frac{1}{2}$. Since $\left(\frac{1}{2}\right) = \frac{4}{5}$, we have

$$\phi\left(\frac{1}{2}, \frac{4}{5}\right) = \phi\left(0, \frac{1}{2}\right).$$

Another example is that in Escher's «Circle Limit III», all the (yellow) fish have the same "size" from a hyperbolic point of view.

Hyperbolic “straight line”, a.k.a. geodesic

A geodesic between two point z_1, z_2 is the shortance path connecting them.

We admit the following:

1. The line $(-1, 1)$ is a geodesic: the shortest path between any two points $r_1, r_2 \in (-1, 1)$ is a portion of the line $(-1, 1)$.
2. A geodesic is preserved by Möbius maps. Otherwise said, if γ is the shortest path between z_1, z_2 then the image of γ by τ, i.e. $\tau(\gamma)$, is the shortant path that connects $\tau(z_1)$ and $\tau(z_2)$.

As a consequence, the hyperbolic geodesics in the disk are “arcs of circle” that intersect in a perpendicular manner the boundary $\partial\mathbb{D}$.

Hyperbolic circle

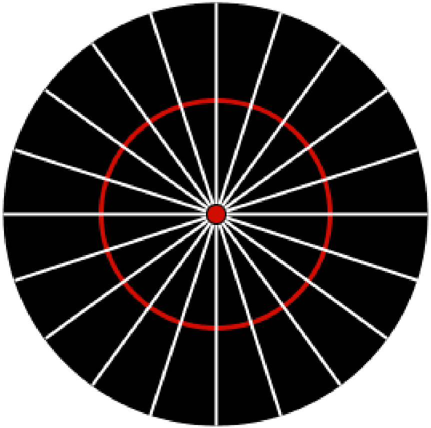
A hyperbolic circle $C(z, \rho)$ of center z and radius $\rho > 0$ is the set of points at a hyperbolic distance $\rho > 0$ with z.

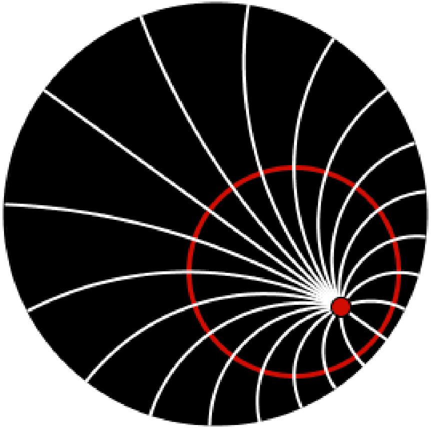
For example, $C(0, \rho)$ looks like a Euclidean circle $C_{\text{Eucl}}(0, r)$ with a difference radius r: the relationship is

$$\rho = \phi(0, r) = 2\text{artanh}(r).$$

Since a Möbius map preseves distances, it also preserves circles: however in general they are "distorted" at the boundary. A hyperbolic circle still look like a Euclidean circle, except that its center is closer to the boundary $\partial\mathbb{D}$.

Hyperbolic circle under Möbius map

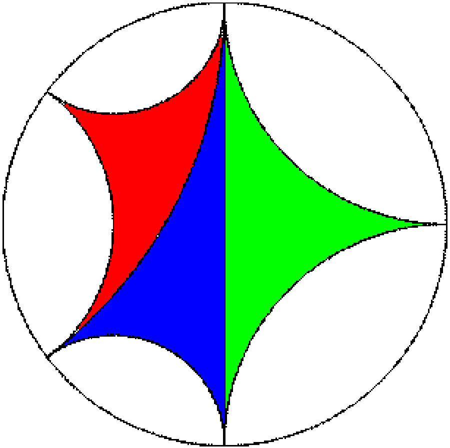




Hyperbolic triangles

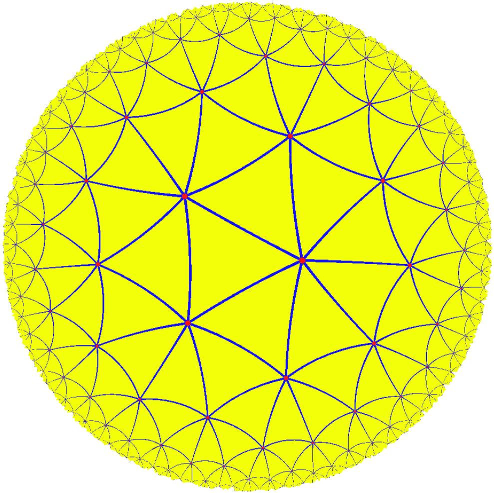
A triangle with vertices A, B, C is defined by the three geodesics between them.

In the most extreme cases we can take vertices on the boundary $\partial\mathbb{D}$.



Hyperbolic triangles

Below are equilateral triangles of angle $\frac{2\pi}{7}$: notice that $\frac{2\pi}{7} + \frac{2\pi}{7} + \frac{2\pi}{7} < \pi$!



Hyperbolic angles

Consider a hyperbolic triangle $\triangle ABC$. Consider the hyperbolic geodesics AB and AC and the angle $\angle A$ between their tangents at the point A. Let a be the hyperbolic length of BC. Defined similarly $\angle B, \angle C, b, c$.

A version of the Gauss-Bonnet theorem says:

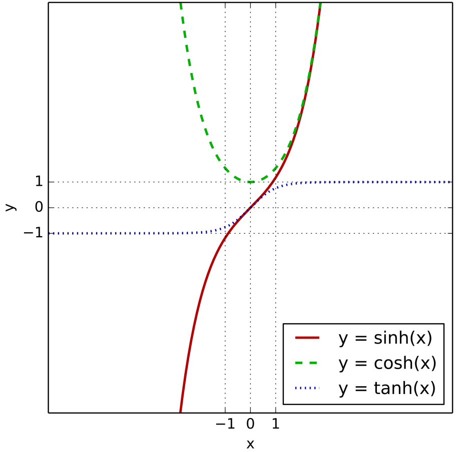
$$\text{Area}(\triangle ABC) = \pi - (A + B + C).$$

So the small the triangle, the more “ordinary” it becomes.

A version of the hyperbolic Pythagoras’ theorem says: if $C = \frac{\pi}{2}$,

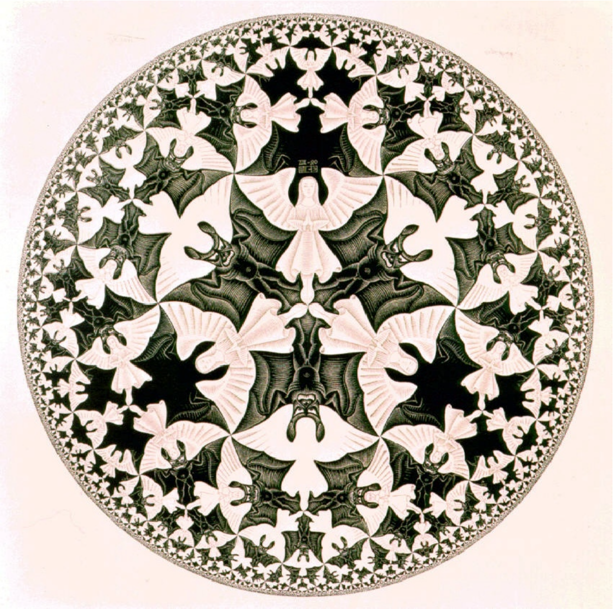
$$\sin(A) = \frac{\sinh(a)}{\sinh(c)}; \quad \cos(A) = \frac{\tanh(b)}{\tanh(c)}; \quad \tan(A) = \frac{\tanh(a)}{\sinh(b)}.$$

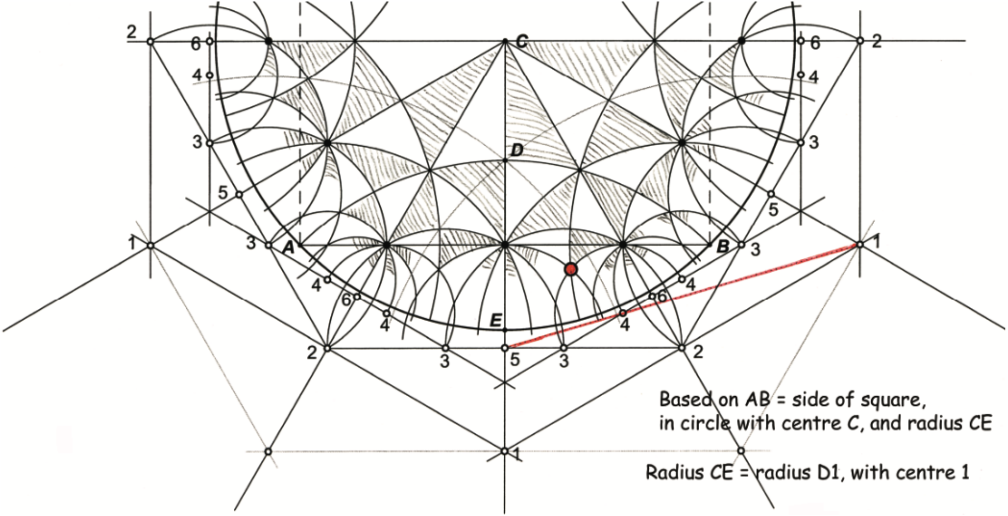
sinh and cosh



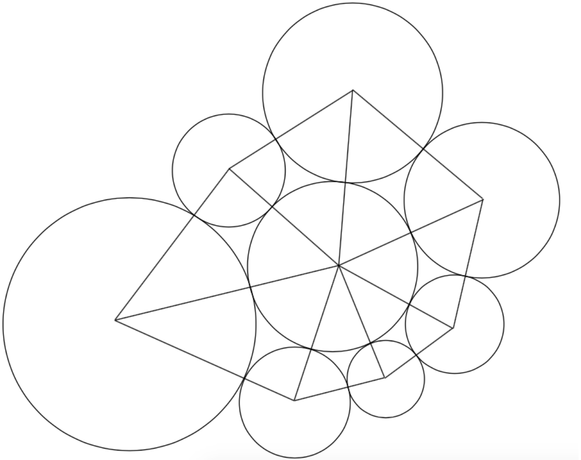
$$\sinh(x) = \frac{e^x - e^{-x}}{2}; \quad \cosh(x) = \frac{e^x + e^{-x}}{2}; \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

Angels and devils, M. C. Escher




Based on AB = side of square,
in circle with centre C, and radius CE
Radius CE = radius D1, with centre 1

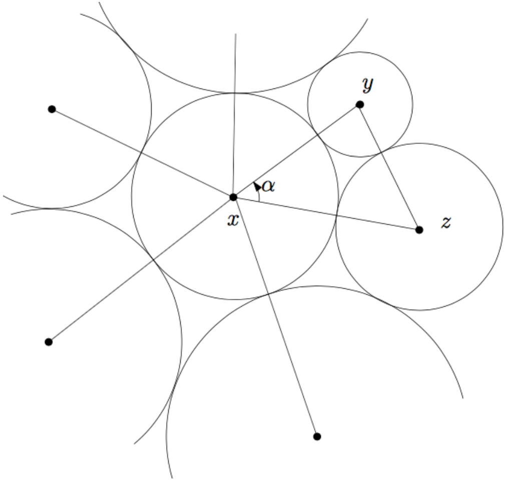
Development: circle packing



Koebe: every planar finite triangulation has a circle packing. Thurston: the circle packing above is rigid (i.e. unique up to Möbius maps and reflections).

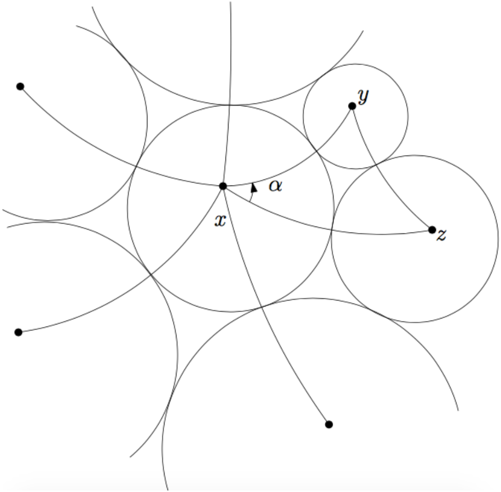
Development: Euclidean circle packing

For a Euclidean circle packing you can calculate angles and radii of circles.



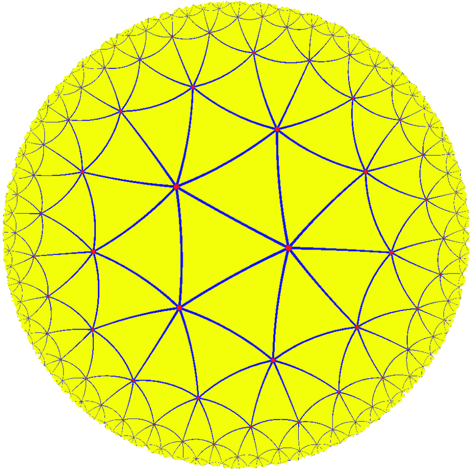
Development: hyperbolic circle packing

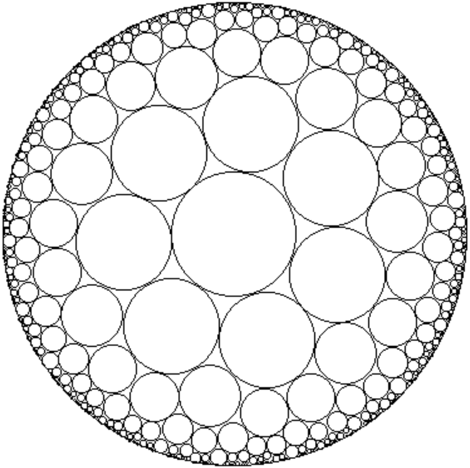
For a hyperbolic circle packing you can also calculate angles and radii of circles!



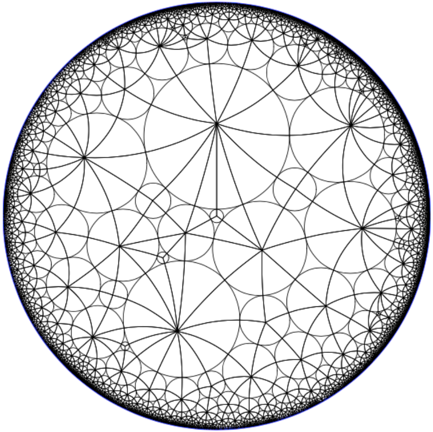
Development: hyperbolic circle packing with degree 7

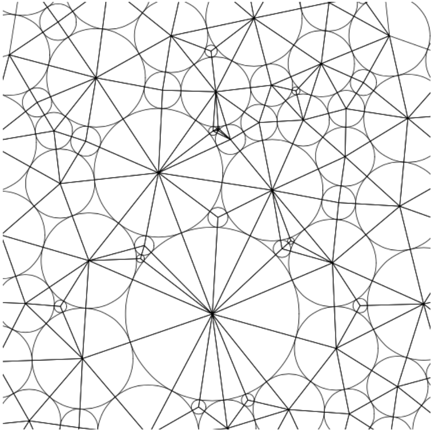
We have seen one of these pictures before :)





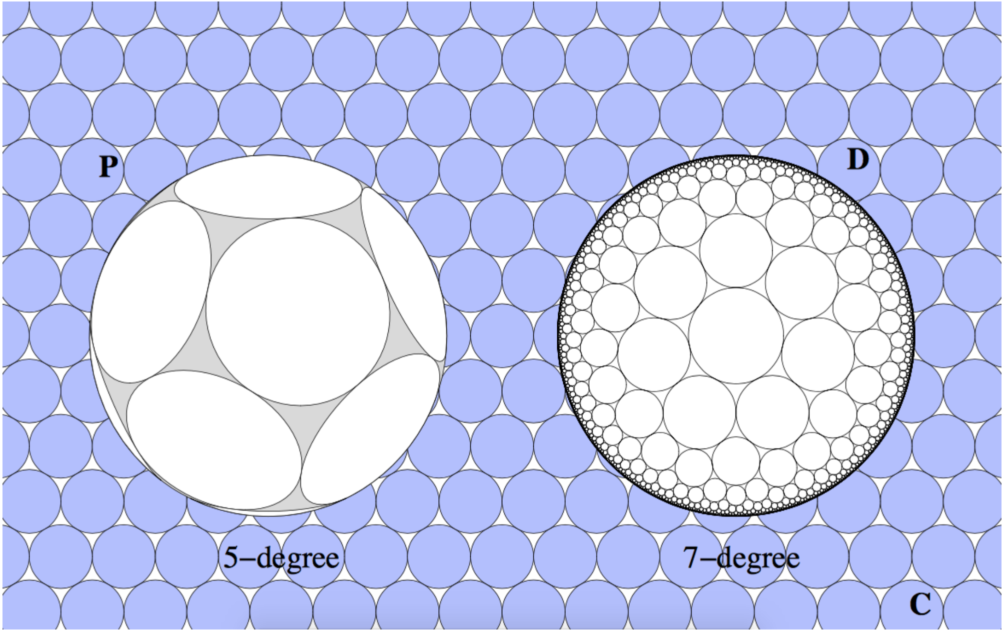
Development: hyperbolic vs Euclidean





He-Schramm: to an infinite planar triangulation, there is a unique associated circle packing, either embedded in $\mathbb{D}$ or in $\mathbb{R}^2$, but not the two at the same time.

Development: Benjamini-Schramm theorem



Development: 6<7

We prove a special case of the Benjamini-Schramm theorem: that the hexagonal circle packing cannnot be embedded in $\mathbb{D}$.

In fact, we will prove that for a circle packing to be embedded in $\mathbb{D}$, it must have average degree $d > 6$.

The basic idea is the following:

- For a Euclidean triangle, the average degree of an angle is $\frac{\pi}{3}$. Thus to fill the angle 2π, you need in average 6 triangles.
- For a hyperbolic triangle, the average degree is stricly smaller than $\frac{\pi}{3}$. Thus to fill the angle 2π, you need in average more thant 6 triangles.

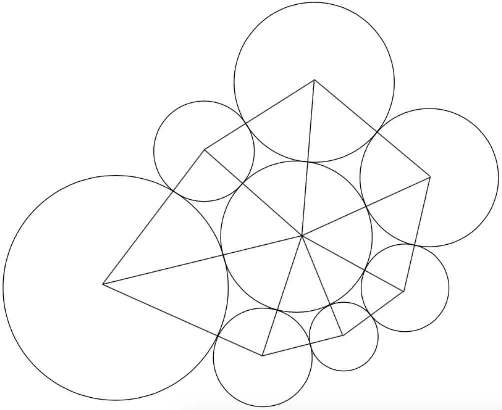
Remember the hyperbolic Gauss-Bonnet: $\text{Area}(\triangle ABC) = \pi - A - B - C$.

Development: transport theory

Consider a finite triangulation with vertices $x_1, \ldots, x_n$. On each vertex x_n we put some weight $c_n > 0$.

A transport of mass is a rule to redistribute the weights c_n. For example, we can define the following procedure: if a vertex has d neighbors, then it gives weight $\frac{c_n}{d}$ to every of its neighbor (in return it will also receive some mass from its neighbors).

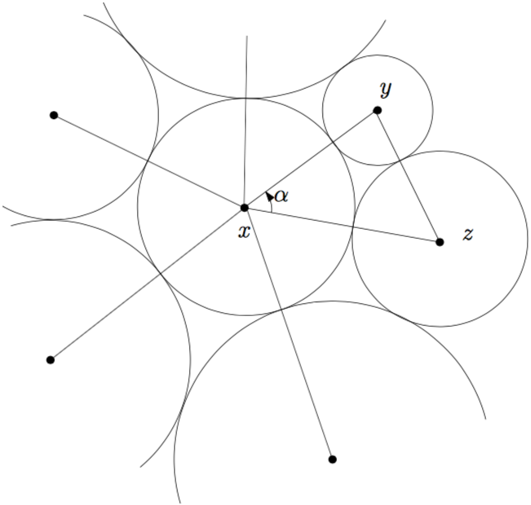
Whatever the rule is, the sum of mass $\sum c_n$ is constant!



Now consider a infinite triangulation.

We say that it is unimodular if for a uniformly randomly chosen vertex, the mass that it gives out is equal to the mass that it receives on average.

For the hexagonal triangulation, every vertex plays the same role and it is “not hard” to verify that it is unimodular.



Benjamini-Schramm's transport: for every vertex A in any $\triangle ABC$ (hyperbolic or not), A gives a mass α equal to the angle $\angle A$ to B and to C.

Any vertex A gives out exactly 4π mass in total (whatever the geometry)!

How much mass does a "uniform" vertex A receive on average? For the hexagonal triangulation: $12 \cdot \mathbb{E}[\angle]$.

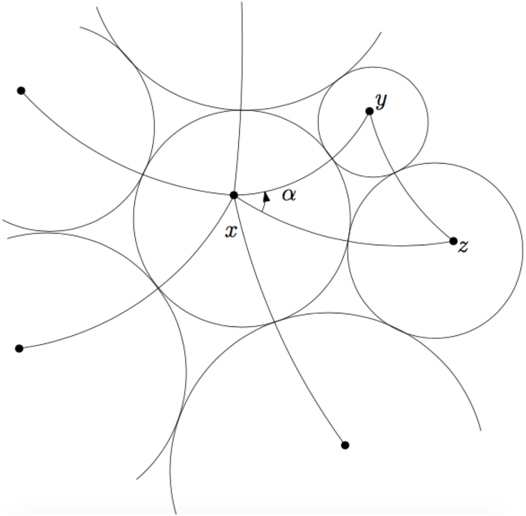


Image credits

- Most of the images on circle packing is by Ken Stephenson.
- Most of the materials is from a lecture note of Asaf Nashmias.